Parabolic Boundary Value Problems With Rough Coefficients

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(*Luke Oliver Dyer*)

Abstract

This thesis is motivated by some of the recent results of the solvability of elliptic PDE in Lipschitz domains and the relationships between the solvability of different boundary value problems. The parabolic setting has received less attention, in part due to the time irreversibility of the equation and difficulties in defining the appropriate analogous time-varying domain. Here we study the solvability of boundary value problems for second order linear parabolic PDE in time-varying domains, prove two main results and clarify the literature on time-varying domains.

The first result shows a relationship between the regularity and Dirichlet boundary value problems for parabolic equations of the form $Lu = \operatorname{div}(A\nabla u) - u_t = 0$ in $\operatorname{Lip}(1, 1/2)$ time-varying cylinders, where the coefficient matrix $A = [a_{ij}(X,t)]$ is uniformly elliptic and bounded. We show that if the Regularity problem $(R)_p$ for the equation $Lu = 0$ is solvable for some $1 < p < \infty$ then the Dirichlet problem $(D^*)_{p'}$ for the adjoint equation $L^*v = 0$ is also solvable, where $p' = p/(p-1)$. This result is analogous to the one established in the elliptic case.

In the second result we prove the solvability of the parabolic L^p Dirichlet boundary value problem for $1 < p \leq \infty$ for a PDE of the form $u_t = \operatorname{div}(A\nabla u) + B \cdot \nabla u$ on time-varying domains where the coefficients $A = [a_{ij}(X,t)]$ and $B = [b_i(X,t)]$ satisfy a small Carleson condition. This result brings the state of affairs in the parabolic setting up to the current elliptic standard. Furthermore, we establish that if the coefficients of the operator A and B satisfy a vanishing Carleson condition, and the time-varying domain is of VMO-type then the parabolic L^p Dirichlet boundary value problem is solvable for all $1 < p \leq \infty$. This is related to elliptic results where the normal of the boundary of the domain is in VMO or near VMO implies the invertibility of certain boundary operators in L^p for all $1 < p < \infty$. This then (using the method of layer potentials) implies solvability of the L^p boundary value problem in the same range for certain elliptic PDE. We do not use the method of layer potentials, since the coefficients we consider are too rough to use this technique but remarkably we recover L^p solvability in the full range of p's as the elliptic case. Moreover, to achieve this result we give new equivalent and localisable definitions of the appropriate time-varying domains.

Lay Summary

The heat equation is a partial differential equation (PDE) which describes how heat flows through an object (via conduction). In this thesis we study generalised versions of the heat equation on materials (or domains) that are allowed to vary in time. This could be an ice cube melting, or a metal bar expanding as it is heated. We look at generalised versions of this equation (parabolic PDE) which, for instance, allow the object to have impurities in it or allow the object to be made from lots of different types of materials, e.g. the earth's crust, a car engine or a glacier.

To solve these equations we require three different pieces of data: how the conduction changes inside the object, the physical shape of the object (and how it changes in time), and how hot the object is on the outside. These three pieces of data are fed into the heat equation and the solution to the heat equation is the heat of the object at any interior point (at any time).

The aim of this thesis is to see just how 'nasty' these three pieces of data can be whilst still allowing us to obtain a reasonable solution at the end. One would then use this theoretical guarantee of a reasonable solution when one models these types of equations on a computer. It would justify that the answer given by a computer is correct.

This equation is not only a model of heat flowing through an object but can model how anything diffuses (spreads out) in time. It has applications to financial mathematical models, mathematical biology and of course the physical sciences.

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Soli Deo gloria

Contents

Chapter 1

Introduction

This thesis studies the following linear second order divergence form parabolic equations on a time-varying domain Ω. It has the form

$$\begin{cases} u_t = \operatorname{div}(A\nabla u) + B \cdot \nabla u & \text{in } \Omega \subset \mathbb{R}^{n+1}, \\ u = f & \text{on } \partial\Omega, \end{cases} \tag{1.0.1}$$

where $A = A(X,t)$ is a $n \times n$ matrix and $B = B(X,t)$ is a $1 \times n$ vector. We assume that A is uniformly elliptic and bounded, that is that there exists positive constants λ and Λ such that

$$\lambda|\xi|^2 \leq \sum_{i,j} a_{ij}(X,t)\xi_i\xi_j \leq \Lambda|\xi|^2 \tag{1.0.2}$$

for almost every $(X,t) \in \Omega$ and all $\xi \in \mathbb{R}^n$. We do not assume any symmetry on A. Furthermore, we usually assume B is locally bounded and satisfies the condition

$$\delta(X,t)|B(X,t)| \leq K \tag{1.0.3}$$

for some uniform constant $K > 0$, where $\delta(X,t)$ is the parabolic distance of a point (X,t) to the boundary $\partial\Omega$. The term $\operatorname{div}(A\nabla u)$ is called the diffusion term and the term $B \cdot \nabla u$ is called the drift term. If we take $B \equiv 0$ then the adjoint of (1.0.1) is

$$\begin{cases} -u_t = \operatorname{div}(A^*\nabla u) & \text{in } \Omega \subset \mathbb{R}^{n+1}, \\ \quad u = f & \text{on } \partial\Omega, \end{cases} \tag{1.0.4}$$

where A^* is the transpose of A.

In this thesis we prove two main results regarding the solvability of (1.0.1) when $f \in L^p$ (theorems 3.1.1 and 4.1.6), another result finding equivalent conditions on the correct domain to study this PDE (theorem 4.2.7), and finally we localise one of these conditions (theorem 4.2.13).

The first result, theorem 3.1.1, proves that if the Regularity problem (R_p) for the operator

$$L = \operatorname{div}(A\nabla) - \partial_t$$

on a $\operatorname{Lip}(1, 1/2)$ cylinder Ω is solvable for some $1 < p < \infty$ then the Dirichlet problem $(D^*)_{p'}$ for the adjoint operator

$$L^* = \operatorname{div}(A^*\nabla) + \partial_t$$

is also solvable on the domain Ω, where p and p' are Hölder conjugates of each other. The second main result, theorem 4.1.6, states that we can solve the L^p Dirichlet problem for $1 < p \leq \infty$ on an admissible domain if we assume the coefficients satisfy a natural, minimal smoothness condition (a Carleson condition).

Motivation for studying general divergence form equations We approach the motivation in the elliptic setting since the parabolic setting is much more delicate. Let us assume we're studying the prototypical elliptic equation, Laplace's equation $\Delta u = 0$, on a Lipschitz graph domain $\Omega = \{(x_0, x) \in \mathbb{R} \times \mathbb{R}^n : x_0 > \phi(x)\}$. There are two different pullback mappings that we could construct to transform this problem onto the upper half space $U = \mathbb{R}^{n+1}_+$ and study it there.

The first pullback mapping we could construct is by flattening the boundary by the map $\rho^{-1}(x_0, x) = (x_0 - \phi(x), x)$. This maps the Laplacian to the operator $L = \operatorname{div}(\nabla A)$ in U where the matrix $A(x)$ is merely bounded, measurable and elliptic; it is not necessarily symmetric. However there is one redeeming feature of A: it is independent of the x_0 variable. This means if u is a solution to $Lu = 0$ then so too is $\partial_{x_0} u$. An important result along this direction is the solution to the Kato square root problem [AHLMT02]; see also the work of [HKMP15] and references therein.

The second more useful pullback mapping is the one given by Dahlberg-Kenig-Stein [Dah86]. Let θ be the following approximation to the identity: θ is even, $\theta \in C_0^\infty(\mathbb{R}^n)$ and $\int \theta = 1$; let $\theta_\lambda(x) = \lambda^{-n}\theta(x/\lambda)$. We define the pullback mapping $\rho : U \to \Omega$ by

$$\rho(x_0, x) = (x, cx_0 + \theta_{x_0} * \phi(x)),$$

where c is a constant chosen so that this mapping is bijective. Again under this pullback mapping the operator Δ pulled back to U defines a new operator $L = \operatorname{div}(\nabla A)$ that is elliptic, bounded and measurable. However this time instead of $A(x_0, x)$ being independent of x_0, A satisfies the following properties:

(i) $|\nabla A(x_0, x)| \lesssim 1/x_0$,

(ii) $x_0 |\nabla A(x_0, x)| \, \mathrm{d}x \, \mathrm{d}x_0$ is a Carleson measure,

(see section 2.5 for the definition of Carleson measures).

When we pass to the parabolic setting a similar result holds only for an appropriate modification of the second pullback mapping, see section 4.2.4. Unfortunately, if we attempt to use the first pullback mapping (flattening) for a time-varying domain which is say Lipschitz space and $\mathrm{Lip}_{1/2}$ in time then the mapping $(x_0, x, t) \to (x_0 - \phi(x, t), x, t)$ does not produce a well defined PDE, see section 4.2.4 for details. For parabolic results when A is independent in the x_0 direction see the papers [AEN16; Nys16; Nys17].

History It has been observed via the method of layer potentials that when the domain on which we consider certain boundary value problems for elliptic or parabolic PDE is sufficiently smooth the question of L^p invertibility of certain boundary operators can be resolved. This can be done using the Fredholm theory since this operator is just a compact perturbation of the identity. This observation then implies the invertibility of this boundary operator for all $1 < p \leq \infty$ and hence solvability of the corresponding L^p boundary value problem in this range.

The notion of how smooth the domain has to be for the above observation to hold has evolved. Initial results for constant coefficient elliptic PDE required domains of at least $C^{1,\alpha}$ type. This was reduced to C^1 domains in an important paper of Fabes, Jodeit and Rivière [FJR78]. Later the method of layer potentials was adapted to variable coefficient settings and the results were extended to elliptic PDE with variable coefficients [Din08] on C^1 domains.

Further progress was made after advancements in singular integrals theory on sets that are not necessary of graph-type [Sem91; HMT10]. It turns out that compactness of the mentioned boundary operator only requires that the normal (which must be well defined at almost every boundary point) belongs to VMO.

This observation for the Stokes system was made in [MMS09] where boundary value problems for domains whose normal belongs to VMO (or is near to VMO in the BMO norm) were considered. In [HMT15] symbol calculus for operators of layer potential type on surfaces with VMO normals was developed and applied to various elliptic PDE including elliptic systems.

So far we have only mentioned elliptic results. One of the first results for the heat equation in Lipschitz cylinders is by Brown [Bro89b]. Here the domain considered is time independent and Fourier methods in the time variable are used. Domains of time-varying type for the heat operator were first considered in the papers [LS88; LM92; LM95; HL96] and again the method of layer potentials was used to establish L^2 solvability. The question of solvability of various boundary value problems for parabolic PDE on time-varying domains has a long history. Recall that in the elliptic setting Dahlberg [Dah77] showed in a Lipschitz domain that the harmonic measure and surface measure are mutually absolutely continuous, and that the elliptic Dirichlet problem is solvable with data in L^2 with respect to the surface measure. Furthermore if the domain is C^1 then the Dirichlet problem is solvable for L^p data for all $1 < p \leq \infty$ [Dah79].

R. Hunt then asked whether Dalhberg's L^2 solvability result held for the heat equation in domains whose boundaries are given locally as functions $\phi(x,t)$, Lipschitz in the spatial variable. Due to the parabolic scaling of the PDE it was conjectured that the correct regularity of $\phi(x,t)$ should be Hölder continuous of order $1/2$ in the time variable t and Lipschitz in the spatial variables x. It turns out that under this assumption the parabolic measure associated with the equation (1.0.1) is doubling [Nys97].

However, in order to answer R. Hunt's question positively one has to consider more regular classes of domains than the one described above. This follows from the counterexample of Kaufman and Wu [KW88]. In that paper it was shown that under just the $\text{Lip}(1,1/2)$ condition on the domain Ω the associated caloric measure (that is the measure associated with the operator $\partial_t - \Delta$) might not be mutually absolutely continuous with the natural surface measure (see theorem 4.2.1 and section 4.2.1 for precise and stronger statements). The issue was resolved in [LM95] where they established that the mutual absolute continuity of the caloric measure and a certain parabolic analogue of the surface measure holds when $1/2$ a time derivative of ϕ is in parabolic $\text{BMO}(\mathbb{R}^n)$ (denoted $D^t_{1/2}\phi \in \text{BMO}$). This is a slightly stronger condition than $\text{Lip}(1,1/2)$ but weaker than $\text{Lip}(1,1/2+\varepsilon)$. We refer to such domains as being of Lewis-Murray type and we call this condition the Lewis-Murray condition. In section 4.2.1 we discuss why these domains are the correct parabolic analogue of Lipschitz domains, at least from a layer potential point of view. Hofmann and Lewis [HL96] subsequently showed that the Lewis-Murray condition is sharp. We thoroughly discuss these domains in section 4.2.

Further work was done by [HL01; Riv03; Riv14] in graph domains and time-varying cylinders satisfying the Lewis-Murray condition where they proved the L^p Dirichlet problem was solvable for all $p > p'$ for some potentially very large p' (due to the technique used there is no control on the size of p'). Finally [DH18] established L^p solvability $2 \leq p \leq \infty$ in domains that are of Lewis-Murray type under a small Carleson condition. We review these and related results further in section 4.1.

While researching the literature on domains of Lewis-Murray type and ways this concept can be localised (in the time variable the half-derivative is a non-local operator and hence any condition imposed on it is difficult to localise) we have realized that important results we planned to rely on are incorrect (see in particular remark 4.2.11). Therefore, section 4.2 sets the literature record straight and more importantly explains in detail the concept of localised domains of Lewis-Murray type.

Main results The first main result of this thesis, theorem 3.1.1, proves that solvability of the regularity problem (R_p) implies solvability of the adjoint Dirichlet problem $(D^*_{p'})$. $(D^*_{p'})$ has boundary data in $L^{p'}(\partial\Omega)$ and (R_p) has boundary data in a Sobolev space $L^p_{1,1/2}(\partial\Omega)$, which is a space of functions with spatial derivatives and a half-time derivative in L^p. See definitions 2.4.6, 2.4.7 and 3.2.1 for the precise definitions of $(D)_p$, $(R)_p$ and $L^p_{1,1/2}$ spaces. We have set $B \equiv 0$ for this result since the adjoint with drift terms is particularly nasty. The domain Ω in which we work here is a $\text{Lip}(1,1/2)$ cylinder — the boundary is given locally as a graph of a function $\phi(x,t)$ which is Lipschitz in the spatial directions and $\text{Lip}_{1/2}$ in time.

Observe that the adjoint L^* is a backward in time parabolic operator. This however does not causes any issues as by the change of variables of $v(X,t) = u(X,-t)$ and $\tilde{A}(X,-t) = A(X,t)$ we see that $L^*u = 0$ on Ω is equivalent to

$$\tilde{L}v = \text{div}(\tilde{A}^*\nabla v) - v_t = 0 \qquad \text{on } \tilde{\Omega}.$$

Here $\tilde{\Omega}$ is the reflection of Ω in the t variable i.e. $\tilde{\Omega} = \{(X,-t) : (X,t) \in \Omega\}$. Hence, the solvability of the $L^{p'}$ Dirichlet problem for the operator L^* on Ω is equivalent to the solvability of the $L^{p'}$ Dirichlet problem for the operator $\tilde{L}$ on $\tilde{\Omega}$; here $\tilde{L}v = 0$ is the usual forward in time parabolic PDE.

This result is motivated by the analogous result in the elliptic setting by [KP93] where, amongst other relationships, they show that (R_p) implied $(D^*)_{p'}$ for elliptic operators $\operatorname{div}(A\nabla\cdot)$ in bounded Lipschitz domains. This has been observed for some specific parabolic PDE (such as the heat equation and constant coefficient systems, [HL96, p. 418; Nys06] respectively) in the smaller class of Lewis-Murray type domains. Nyström [Nys06] also shows that no duality can be expected between the Dirichlet and Neumann boundary value problems in non-smooth time-varying domains.

In the second result, theorem 4.1.6, of this thesis we establish an L^p solvability result for parabolic PDE on time-varying cylinders satisfying the Lewis-Murray condition in the full range $1 < p \leq \infty$. We assume the coefficients A and B satisfy a natural, minimal smoothness condition, called a Carleson condition, see (4.1.6). This generalises the result of [DH18] to below $p = 2$. Furthermore, if the coefficients satisfy a vanishing Carleson condition and the domain is of VMO-type then we show the solvability of the L^p Dirichlet problem for all $1 < p \leq \infty$.

Moreover, by using our equivalent conditions on the domain and our localisation result (theorems 4.2.7 and 4.2.13) we locally define a domain by truly local graphs called an admissible domain (definition 4.2.20). These domains are equivalent to Lewis-Murray cylinders, in the sense that a domain is a Lewis-Murray cylinder if and only if it is an admissible domain. However, an advantage of this definition (apart from it being constructed by local functions) is the much more nuanced control over the norms of the local graphs — they are allowed to have large $\mathrm{Lip}(1, 1/2)$ constants as long as the appropriate BMO norms are small. Given a domain Ω and a parabolic operator L this allows us to deduce the solvability of the L^p Dirichlet problem for a much larger range of p, see remark 4.2.25 for details. The reason why we have to introduce both Lewis-Murray cylinders and admissible domains (definitions 4.2.3 and 4.2.20 respectively) is because the original attempts to localise $D^t_{1/2}\phi \in \mathrm{BMO}$, a non-local operator, in [DH18; DPP17] for Lewis-Murray cylinders is not obvious and is thought not to hold. This is due to both the cancellation that occurs within the non-truncated operator and the BMO norm being strongly influenced by cancellation.

The coefficients we consider are very rough and in particular the method of layer potentials cannot be used. Despite this we recover (in the parabolic setting) an analogue of [MMS09; HMT15]. Remarkably when the coefficients satisfy a vanishing Carleson condition and the domain is of VMO-type we recover the full range of solvability that holds for smooth coefficients (via the layer potential method).

Our proof is however completely different from the layer potential method; for example, at no point is compactness used. The proof is also substantially different than the case $2 \leq p \leq \infty$ of [DH18] in the following way. We were inspired by [DPP07] and have used a so called p-adapted square function in order to prove L^p solvability. However, due to the presence of the parabolic term a second type of square function will arise, namely

$$\int_\Omega |u_t(X,t)|^2 |u(X,t)|^{p-2} \delta(X,t)^3 \,\mathrm{d}X \,\mathrm{d}t, \tag{1.0.5}$$

where $\delta(X, t)$ is the parabolic distance to the boundary. When $p = 2$ such object is called the 'area function' in [DH18] and there it is shown that it can be dominated by the usual square function. It turns out that the case $1 < p < 2$ is substantially more complicated and we were only able to establish required bounds for (1.0.5) for non-negative u after a substantial effort by proving p-adapted version of a Caccioppoli inequality for the second gradient (proposition 4.4.9).

There is also an issue of whether the p-adapted square function is actually well-defined and locally finite (as the exponent on $|u|$ is negative). We prove that when u is a solution of a parabolic PDE the p-adapted square function is indeed well defined by adapting a recent regularity result from [DP16] and using that we can bound the p-adapted square function by the non-tangential maximal function. The paper [DP16] deals with complex coefficient elliptic PDE but the method used there can be adapted to the parabolic setting; see theorem 4.4.3 and remark 4.5.3 for details.

Many results in the parabolic setting are motivated by previous results in the elliptic setting and ours is no different. Let us briefly overview the major elliptic results related to theorem 4.1.6.

The papers [KKPT00; KP01] started the study of non-symmetric divergence form elliptic operators with bounded and measurable coefficients. Kenig and Pipher [KP01] used [KKPT00] to show that the elliptic measure of operators satisfying a type of Carleson measure condition is in A_∞ and hence the L^p Dirichlet problem is solvable for some p, potentially large. In [DPP07], the authors improved the result of [KP01] in the following way. They showed if

$$\delta(X)^{-1}\left(\operatorname*{osc}_{B_{\delta(X)/2}(X)} a_{ij}\right)^2 \quad \text{and} \quad \delta(X)\left(\sup_{B_{\delta(X)/2}(X)} b_i\right)^2 \tag{1.0.6}$$

are densities of Carleson measures with vanishing Carleson norms then the L^p Dirichlet problem is solvable for all $1 < p \leq \infty$. A similar result for the elliptic Neumann and regularity boundary value problem is established in [DPR17].

The parabolic analogue of the elliptic Carleson condition (1.0.6) is that

$$\delta(X,t)^{-1}\sup_{i,j}\left(\operatorname*{osc}_{B_{\delta(X,t)/2}(X,t)} a_{ij}\right)^2 + \delta(X,t)\left(\sup_{B_{\delta(X,t)/2}(X,t)} b_i\right)^2 \tag{1.0.7}$$

is the density of a Carleson measure on Ω with a small Carleson norm, where again $\delta(X,t)$ is the parabolic distance of a point (X,t) to the boundary $\partial\Omega$.

The condition (1.0.7) arises naturally as described previously for the elliptic setting. Let $\Omega = \{(x_0,x,t) : x_0 > \phi(x,t)\}$ for a function ϕ which satisfies the Lewis-Murray condition above. Let $\rho : U \to \Omega$ be a mapping from the upper half space U to Ω. Consider $v = u \circ \rho$. It will follow that if u solves (1.0.1) in Ω then v will be a solution to a parabolic PDE similar to (1.0.1) in U. In particular if ρ is chosen to be the mapping in (4.2.53) then the coefficients of the new PDE for v will satisfy a Carleson condition like (1.0.7), c.f. lemma 4.2.26, provided the original coefficients (for u) were either smooth or constant.

Furthermore, if we do not insist on control over the size of the Carleson norm then we can still infer solvability of the L^p Dirichlet problem for large p, as in [HL01; Riv03; Riv14].

1.1 Layout and Aims

Chapter 2 Here we discuss the relevant background material that is used in the following chapters and we give context to our results. We review well known estimates and results for solutions to (1.0.1) and its adjoint in $\mathrm{Lip}(1,1/2)$ domains. We examine the continuous Dirichlet problem and introduce the parabolic measure in section 2.3. After this, we explore A_p weights and its implications to the solvability of the L^p Dirichlet problem. Finally, we give a quick overview of pertinent harmonic analysis results used in later chapters: namely Carleson measures, BMO, H^1, Calderòn-Zygmund operators and a few of the many intricate relationships linking these topics.

Chapter 3 In this chapter we first review parabolic Sobolev spaces on the whole space, give a definition of them on domains and prove the consistency of this definition. After introducing a Poincaré type inequality, we prove $(R)_p$ implies $(D^*)_{p'}$ using the strategy from the elliptic proof in [KP93] as a guide. The results from this chapter appear in a condensed form in [DD17].

Chapter 4 We start chapter 4 by giving a brief survey of the known L^p Dirichlet results in time-varying domains. One of the largest sections in this chapter, section 4.2, focuses on investigating the Lewis-Murray condition. We begin by motivating this condition from the perspective of layer potentials and review the known equivalent conditions for $D^t_{1/2}\phi \in \mathrm{BMO}$ if ϕ is $\mathrm{Lip}(1,1/2)$. Following this we find three equivalent conditions (one already known with a questionable proof) with equivalence of norms to $\|\mathbb{D}\phi\|_*$. These results and the proofs behind them (with inspiration from [Str80]) may be of independent interest especially in the setting of parabolic PDE in time-varying domains or parabolic uniform rectifiability. We further prove that one of these conditions is localisable. Finally we are able to state our definition of admissible and VMO-type domains, and we compare them to the definition of Lewis-Murray cylinders. We define our pullback mapping from Ω to the upper half space and after a modification of a proof from [HL96], we show how under this transformation a Carleson condition on our coefficients is preserved, c.f. (1.0.7).

After showing that the basic inequalities from section 2.2 still hold in our domain and with small drift terms, we move onto some delicate arguments showing that the p-adapted square and p-adapted area functions are well defined (the integrals are a priori not locally integrable); and that we can bound the p-adapted area function by the p-adapted square function. Subsequently we can begin to prove the solvability of the L^p Dirichlet problem, theorem 4.1.6. We do this via the standard non-tangential maximal and square function arguments and, although there are a few new terms to deal with when we for instance integrate by parts, all the heavy lifting has already been done.

The results from this chapter appear in a shortened form in [DDH18].

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Chapter 2

Background Material

In this chapter we review some well known material about second order linear parabolic PDE that is used throughout the thesis. In section 2.1 we define the most general class of domains, $\mathrm{Lip}(1, 1/2)$ domains, that our parabolic PDE are studied on. Section 2.2 reviews some classical inequalities and results of solutions to parabolic operators. In section 2.3 we recall the continuous Dirichlet problem, study its solvability in $\mathrm{Lip}(1, 1/2)$ domains, and introduce the parabolic measure and the Green's function which are needed for chapter 3. Section 2.4 defines the L^p Dirichlet problem, and studies the consequences and implications of the solvability of this problem. It includes a detailed subsection on weights, a brief overview of the kernel function and a couple of perturbation results. Section 2.5 introduces a number of harmonic analysis objects and results that we use throughout this thesis: BMO, H^1, Carleson measures, parabolic Calderòn-Zygmund operators as well as key relationships between these topics. Apart from lemma 2.3.5 all the results in this chapter are already known.

2.1 Parabolic Operators and $\mathrm{Lip}(1, 1/2)$ Cylinders

Here and throughout we consistently use ∇ to denote the gradient in the spatial variables, u_t or ∂_t the gradient in the time variable and use $Du = (\nabla u, \partial_t u)$ for the full gradient of u. We write $f \lesssim g$ to mean that there exists a positive constant c such that $f \leq cg$, we write $f \lesssim_\varepsilon g$ to mean c depends on ε, and $f \sim g$ to mean $f \lesssim g$ and $g \lesssim f$.

Recall we are studying solutions to the following parabolic problem

$$\begin{cases} u_t = \mathrm{div}(A\nabla u) + B \cdot \nabla u & \text{in } \Omega \subset \mathbb{R}^{n+1}, \\ u = f & \text{on } \partial\Omega. \end{cases} \tag{2.1.1}$$

It has the following adjoint when $B \equiv 0$

$$\begin{cases} -u_t = \mathrm{div}(A^*\nabla u) & \text{in } \Omega \subset \mathbb{R}^{n+1}, \\ \quad u = f & \text{on } \partial\Omega. \end{cases} \tag{2.1.2}$$

We define a weak solution to our parabolic operator as follows.

Definition 2.1.1 ([Aro68]). *We say that u is a weak solution to a parabolic operator of the form (2.1.1) in Ω if $u, \nabla u \in L^2_{\mathrm{loc}}(\Omega)$, $\sup_\tau \|u(\cdot, \tau)\|_{L^2_{\mathrm{loc}}(\Omega_\tau)} < \infty$ and*

$$\int_\Omega -u\psi_t + A\nabla u \cdot \nabla\psi - B \cdot \nabla u \psi \,\mathrm{d}X\,\mathrm{d}t = 0 \tag{2.1.3}$$

for all $\psi \in C_0^\infty(\Omega)$. A weak solution to the adjoint equation (2.1.2) is defined similarly if

$$\int_\Omega u\psi_t + A^*\nabla u \cdot \nabla\psi \,\mathrm{d}X\,\mathrm{d}t = 0 \tag{2.1.4}$$

for all $\psi \in C_0^\infty(\Omega)$.

We now define the class of $\mathrm{Lip}(1, 1/2)$ time-varying cylinders in [Kem72; Bro89a] whose boundaries are given locally as functions $\phi(x,t)$ which are Lipschitz in the spatial variables and $\mathrm{Lip}_{1/2}$ in the time variable. At each time $\tau \in \mathbb{R}$ the set of points in Ω with fixed time $t = \tau$, that is $\Omega_\tau = \Omega \cap \{t = \tau\}$, is a non-empty bounded Lipschitz domain in $\mathbb{R}^n$. We start with a few preliminary definitions, motivated by the standard definition of a Lipschitz domain.

Definition 2.1.2 (ℓ-cylinder)**.** $\mathbb{Z} \subset \mathbb{R}^n \times \mathbb{R}$ *is an* ℓ*-cylinder of diameter* d *if there exists a coordinate system* $(x_0, x, t) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R}$ *obtained from the original coordinate system only by translation in the spatial and time variables and rotation in the spatial variables only such that*

$$\mathbb{Z} = \{(x_0, x, t) : |x| < d, |t| < d^2, |x_0| < 2n(\ell + 1)d\}$$

and for $s > 0$

$$s\mathbb{Z} := \{(x_0, x, t) : |x| < sd, |t| < s^2 d^2, |x_0| < 2n(\ell + 1)sd\}.$$

We define a *parabolic boundary cube* in $\mathbb{R}^{n-1} \times \mathbb{R}$, for a constant $r > 0$, as

$$Q_r(x,t) = \{(y,s) \in \mathbb{R}^{n-1} \times \mathbb{R} : |x_i - y_i| < r \text{ for all } 1 \le i \le n-1,\ |t-s|^{1/2} < r\}. \tag{2.1.5}$$

Definition 2.1.3. $\Omega \subset \mathbb{R}^n \times \mathbb{R}$ *is a* $\mathrm{Lip}(1, 1/2)$ *cylinder with character* (ℓ, N, d) *if for any time* $\tau \in \mathbb{R}$ *there are at most* N ℓ*-cylinders* $\{\mathbb{Z}_j\}_{j=1}^N$ *of diameter* d *satisfying the following conditions:*

(1) $\partial\Omega \cap \{|t - \tau| \le d^2\} = \bigcup_j^N (\mathbb{Z}_j \cap \partial\Omega)$.

(2) In the coordinate system (x_0, x, t) *of the* ℓ*-cylinder* $\mathbb{Z}_j$

$$\mathbb{Z}_j \cap \Omega \supset \left\{(x_0, x, t) \in \Omega : |x| < d, |t| < d^2, \delta(x_0, x, t) \le \frac{d}{2}\right\}.$$

(3) $8\mathbb{Z}_j \cap \partial\Omega$ *is the graph* $\{x_0 = \phi_j(x,t)\}$ *of a function* $\phi_j : Q_{8d} \to \mathbb{R}$*, with* $Q_{8d} \subset \mathbb{R}^{n-1} \times \mathbb{R}$*, such that*

$$|\phi_j(x,t) - \phi_j(y,s)| \le \ell \left(|x - y| + |t - s|^{1/2}\right) \text{ and } \phi_j(0,0) = 0. \tag{2.1.6}$$

Here and throughout $\delta(x_0, x, t) := \mathrm{dist}\left((x_0, x, t), \partial\Omega\right)$ and dist is the *parabolic distance*, $\mathrm{dist}[(X,t),(Y,s)] = |X - Y| + |t - s|^{1/2}$. The *parabolic norm* $\|(X,t)\|$ on $\mathbb{R}^n \times \mathbb{R}$ is defined as the unique positive solution ρ to the following equation

$$\frac{|X|^2}{\rho^2} + \frac{t^2}{\rho^4} = 1. \tag{2.1.7}$$

One can show that $\|(X,t)\| \sim |X| + |t|^{1/2}$ and that this norm scales correctly according to the parabolic nature of the PDE. If we attach this norm to $\mathbb{R}^n \times \mathbb{R}$ then this is a space of homogeneous type with the following polar decomposition

$$\begin{aligned} (X,t) &= (\rho\theta_0, \dots, \rho\theta_{n-1}, \rho\theta_n), \\ \mathrm{d}X\mathrm{d}t &= \rho^n(1 + \theta_n^2)\,\mathrm{d}\rho\,\mathrm{d}\theta, \end{aligned} \tag{2.1.8}$$

where $\theta \in \mathbb{S}^n$ and $\mathrm{d}\theta$ is the surface area of the unit sphere.

Remark 2.1.4. It follows from this definition that for each $\tau \in \mathbb{R}$ the time-slice $\Omega_\tau = \Omega \cap \{t = \tau\}$ of a $\mathrm{Lip}(1, 1/2)$ cylinder $\Omega \subset \mathbb{R}^n \times \mathbb{R}$ is a bounded Lipschitz domain in $\mathbb{R}^n$ with character (ℓ, N, d). Therefore, the Lipschitz domains Ω_τ for all $\tau \in \mathbb{R}$ all have uniformly bounded diameter. That is

$$\inf_{\tau \in \mathbb{R}} \mathrm{diam}(\Omega_\tau) \sim d \sim \sup_{\tau \in \mathbb{R}} \mathrm{diam}(\Omega_\tau),$$

where the implied constants in the estimate only depend on N. In particular, if $\mathcal{O} \subset \mathbb{R}^n$ is a bounded Lipschitz domain then the parabolic cylinder $\mathcal{O} \times \mathbb{R}$ is an example of a domain satisfying definition 2.1.3.

Definition 2.1.5 (Pullback transformation)**.** *Let Ω be a* $\mathrm{Lip}(1,1/2)$ *cylinder with character (ℓ, N, d) then we define the pullback transformation $\rho_j : Q_{8d} \to \partial\Omega \cap 8\mathbb{Z}_j$, with $Q_{8d} \subset \mathbb{R}^{n-1} \times \mathbb{R}$, to be*

$$\rho_j(x,t) = (\phi_j(x,t), x, t).$$

This mapping transforms a set on the boundary of the upper half space into a subset of $\partial\Omega$. By condition (1) of definition 2.1.3, $\partial\Omega \cap \{|t-\tau| \leq d^2\}$ can be fully described by at most N pullback transformations ρ_j.

Remark 2.1.6. By multiplying ϕ_j with a suitable cut off function we may assume ϕ_j is defined on $\mathbb{R}^{n-1} \times \mathbb{R}$, ρ_j is defined on $\mathbb{R}^{n-1} \times \mathbb{R}$ with comparable $\mathrm{Lip}(1,1/2)$ norms, and all the axioms of definition 2.1.3 hold with $8\mathbb{Z}_j$ replaced by $4\mathbb{Z}_j$.

We consider solvability of the L^p Dirichlet and L^p regularity boundary value problems with respect to the following surface measure σ.

Definition 2.1.7. *Let $\Omega \subset \mathbb{R}^n \times \mathbb{R}$ be a* $\mathrm{Lip}(1,1/2)$ *cylinder with character (ℓ, N, d). We define the measure σ on sets $A \subset \partial\Omega$ to be*

$$\sigma(A) = \int_{-\infty}^{\infty} \mathscr{H}^{n-1}\left(A \cap \{(X,t) \in \partial\Omega\}\right) \mathrm{d}t, \tag{2.1.9}$$

where $\mathscr{H}^{n-1}$ is the $n-1$ dimensional Hausdorff measure on the Lipschitz boundary $\partial\Omega_\tau = \partial\Omega \cap \{t = \tau\}$.

This measure σ may not be comparable to the usual surface measure on $\partial\Omega$: in the t-direction the functions ϕ_j from the definition 2.1.3 are only $\mathrm{Lip}_{1/2}$ and hence the standard surface measure might not be locally finite. However, our definition assures that for any $A \subset 8\mathbb{Z}_j$, where $\mathbb{Z}_j$ is an ℓ-cylinder, we have

$$\mathscr{H}^n(A) \sim \sigma\left(\rho_j^{-1}(A)\right), \tag{2.1.10}$$

where the implicit constant in (2.1.10), by which these measures are comparable, only depends on ℓ, the $\mathrm{Lip}(1,1/2)$ norm of the domain Ω. If Ω has a smoother boundary, such as Lipschitz (in all variables) or better, then the measure σ is comparable to the usual n-dimensional Hausdorff measure $\mathscr{H}^n$. In particular, this holds for a parabolic cylinder $\mathcal{O} \times \mathbb{R}$.

Notation We standardise some notation that is used throughout this thesis.

Definition 2.1.8. *Let Ω be a* $\mathrm{Lip}(1,1/2)$ *cylinder from definition 2.1.3. For $(Y,s) \in \partial\Omega$, $(X,t), (Z,\tau) \in \Omega$ and $r > 0$ we write:*

$$\begin{aligned}
B_r(X,t) &= \{(Z,\tau) \in \mathbb{R}^n \times \mathbb{R} : \mathrm{dist}[(X,t),(Z,\tau)] < r\},\\
Q_r(X,t) &= \{(Z,\tau) \in \mathbb{R}^n \times \mathbb{R} : |x_i - z_i| < r \text{ for all } 0 \leq i \leq n-1,\ |t-\tau|^{1/2} < r\},\\
\Psi_r(Y,s) &= \{(Z,\tau) \in \mathbb{R}^n \times \mathbb{R} : |y_0 - z_0| < 2n\ell r,\ |y_i - z_i| < r,\ |t-\tau|^{1/2} < r\},\\
\Delta_r(Y,s) &= \partial\Omega \cap B_r(Y,s), \qquad T(\Delta_r) = \Omega \cap B_r(Y,s),\\
\delta(X,t) &= \inf_{(Y,s)\in\partial\Omega} \mathrm{dist}[(X,t),(Y,s)].
\end{aligned}$$

Note that $\Psi_r(Y,s)$ is a parabolic cube elongated in the x_0 direction so that the graph of ϕ_j does not escape the cube (in the x_0 direction). Sometime we take Q_r to be a parabolic cube on the boundary, as in (2.1.5). When the context is not clear whether $Q_r \subset \mathbb{R}^{n+1}$ or $Q_r \subset \mathbb{R}^{n-1} \times \mathbb{R}$ then we write it explicitly. Furthermore, when $Q_r(Y,s)$ is a parabolic boundary cube the Carleson region in the upper half space is $T(Q_r) = (0,r) \times Q_r$, with an analogous definition in Ω. Sometimes for a cube $Q_r \subset \mathbb{R}^{n+1}$ we decompose it into its spatial and temporal cubes, $Q_r = J_r \times I_r$, where $J_r \subset \mathbb{R}^n$ is a cube in the spatial coordinates and I_r is a (parabolically scaled) interval in time — $I_r = (s - r^2, s + r^2)$.

2.2 Basic Properties

The following properties of solutions are fundamental to the subject and can be found in [Nas58; Aro68; Mos64; Mos71; Fri64]. We have chosen to present them as found in [HL01]. Recall (1.0.2) is the elliptic and bounded property of A, and (1.0.3) is $\delta|B| \leq K$.

Interior Estimates

Lemma 2.2.1 (A Caccioppoli inequality, [Aro68]). *Let A and B satisfy* (1.0.2) *and* (1.0.3) *respectively, and suppose that u is a weak solution of* (2.1.1) *or* (2.1.2) *in $Q_{4r}(X,t) \subset \mathbb{R}^{n+1}$ with $0 < r < \delta(X,t)/8$. Then there exists a constant $C = C(\lambda, \Lambda, n)$ such that*

$$r^n \left(\sup_{Q_{r/2}(X,t)} u \right)^2 \leq C \sup_{t-r^2 \leq s \leq t+r^2} \int_{Q_r(X,t) \cap \{t=s\}} u^2(Y,s)\,\mathrm{d}Y + C \int_{Q_r(X,t)} |\nabla u|^2 \,\mathrm{d}Y\,\mathrm{d}s$$
$$\leq \frac{C^2}{r^2} \int_{Q_{2r}(X,t)} u^2(Y,s)\,\mathrm{d}Y\,\mathrm{d}s.$$

Lemma 2.2.2 (A Caccioppoli inequality for the second gradient, [DH18]). *Let A and B satisfy* (1.0.2) *and* (1.0.3) *respectively, and further let $\delta(X,t)|\nabla A(X,t)| \leq K$. Suppose that u is a weak solution of* (2.1.1) *or* (2.1.2) *in $Q_{4r}(X,t) \subset \mathbb{R}^{n+1}$ with $0 < r < \delta(X,t)/8$. Then there exists a constant $C = C(\lambda, \Lambda, n, K)$ such that*

$$\int_{Q_r(X,t)} |\nabla^2 u|^2 \,\mathrm{d}Y\,\mathrm{d}s \leq \frac{C}{r^2} \int_{Q_{2r}(X,t)} |\nabla u|^2(Y,s)\,\mathrm{d}Y\,\mathrm{d}s.$$

We remarkably prove a p-adapted version of this lemma for non-negative u and $1 < p < 2$ (where $\mathrm{d}Y\,\mathrm{d}s$ is replaced by the weight $u^{p-2}\,\mathrm{d}Y\,\mathrm{d}s$) in proposition 4.4.9. This is the key estimate mentioned in the introduction which allows us to show that the non-tangential maximal function is bounded by the square function in the proof of theorem 4.1.6.

Lemmas 3.4 and 3.5 in [HL01] give the following two estimates for weak solutions of (2.1.1) or (2.1.2).

Lemma 2.2.3 (Interior Hölder continuity). *Let A and B satisfy* (1.0.2) *and* (1.0.3), *and suppose that u is a weak solution of* (2.1.1) *or* (2.1.2) *in $Q_{4r}(X,t)$ with $0 < r < \delta(X,t)/8$. Then for any $(Y,s), (Z,\tau) \in Q_{2r}(X,t)$*

$$|u(Y,s) - u(Z,\tau)| \leq C \left(\frac{|Y-Z| + |s-\tau|^{1/2}}{r} \right)^\alpha \sup_{Q_{4r}(X,t)} |u|,$$

where $C = C(\lambda, \Lambda, n)$, $\alpha = \alpha(\lambda, \Lambda, n)$ and $0 < \alpha < 1$.

One consequence of this lemma is $u \in C(\Omega)$ and so it makes sense to talk about the pointwise value of a solution u at any interior point.

Lemma 2.2.4 (Harnack inequality). *Let A and B satisfy* (1.0.2) *and* (1.0.3), *and suppose that u is a weak non-negative solution of* (2.1.1) *in $Q_{4r}(X,t)$, with $0 < r < \delta(X,t)/8$. Suppose that $(Y,s), (Z,\tau) \in Q_{2r}(X,t)$ then there exists $C = C(\lambda, \Lambda, n)$ such that, for $\tau < s$,*

$$u(Z,\tau) \leq u(Y,s) \exp\left[C \left(\frac{|Y-Z|^2}{|s-\tau|} + 1 \right) \right].$$

If $u \geq 0$ is a weak solution of (2.1.2) *then this inequality holds when $\tau > s$.*

Boundary Estimates

We state a version of the maximum principle from [DH18] that is a modification of [HL01, Lemma 3.38]; see [Fri64] for more details.

Lemma 2.2.5 (Maximum principle)**.** *Let Ω be a* $\mathrm{Lip}(1,1/2)$ *cylinder, A and B satisfy* (1.0.2) *and* (1.0.3)*, and let u and v be bounded continuous weak solutions to* (2.1.1) *in Ω. If $|u|, |v| \to 0$ uniformly as $t \to -\infty$ and*

$$\limsup_{(Y,s)\to(X,t)} (u-v)(Y,s) \leq 0$$

for all $(X,t) \in \partial\Omega$ then $u \leq v$ in Ω. The analogous result holds if u and v are weak solutions to (2.1.2)*. In addition, if $u \leq v$ on the boundary of $\Omega \cap \{t \geq \tau\}$ for a given time τ then the assumption that $|u|, |v| \to 0$ uniformly as $t \to -\infty$ is not necessary.*

Corkscrew points, defined shortly, help to present an important difference between elliptic and parabolic equations — time lag and time irreversibility. As we've seen a preview of in lemma 2.2.4, and we see more of in the next few lemmas, the value of the solution is only affected by its past (and not its future values), and the time it takes for the solution to diffuse.

Definition 2.2.6 (Corkscrew points, [DPP17])**.** *Let Ω be a* $\mathrm{Lip}(1,1/2)$ *cylinder with character (ℓ, N, d). For any surface ball $\Delta_r = \Delta_r(Y,s) \subset \partial\Omega$ with $0 < r \lesssim d$ we say that a point $(X,t) \in \Omega$ is a corkscrew point of the ball Δ_r if*

$$t = s + 2r^2 \quad \textit{and} \quad \delta(X,t) \sim r \sim \mathrm{dist}[(X,t),(Y,s)].$$

That is, the point (X,t) is an interior point of Ω of distance to the ball Δ_r and the boundary $\partial\Omega$ of order r. The point (X,t) also lies at of order r^2 later in time than the ball Δ_r. Finally, the implied constants in the definition only depend on the domain Ω but not on r and the point (Y,s).

Each ball of radius $0 < r \lesssim d$ has infinitely many corkscrew points. For each ball we choose one and denote it by $V(\Delta_r)$ or V_r, if there is no confusion as to which ball the corkscrew point belongs to. We define corkscrew points $V(\Psi_r)$ of Ψ_r in the same way. After fixing a domain, the corkscrew points of Δ_r and Ψ_r are equivalent.

When considering the adjoint equation (2.1.2) *we need to use backwards in time corkscrew points, which we denote as V_r^-, and are at a point in time $t = s - 2r^2$.*

Remark 2.2.7. Given that the time slices Ω_τ of the domain Ω are of approximately diameter d the corkscrew points do not exists for balls of sizes $r \gtrsim d$.

The following Carleson type estimate was proved for Lipschitz cylinders in [Sal81] and extended to $\mathrm{Lip}(1,1/2)$ cylinders in [Nys97, Lemma 2.3].

Lemma 2.2.8 (Carleson type estimate)**.** *Let Ω be a* $\mathrm{Lip}(1,1/2)$ *cylinder with character (ℓ, N, d), A satisfy* (1.0.2) *and $B \equiv 0$. Let u be a non-negative weak solution of* (2.1.1) *or the adjoint* (2.1.2) *in $\Psi_{2r}(Y,s)$ with $(Y,s) \in \partial\Omega$ and $0 < r < d/2$. Let u vanish continuously on $\Psi_{2r}(Y,s) \cap \partial\Omega$ then there exists $C = C(\ell, \lambda, \Lambda, n)$ such that for all $(X,t) \in \Psi_r(Y,s)$*

$$u(X,t) \leq Cu(V_r^{\pm}), \tag{2.2.1}$$

where the plus sign is taken when u is a weak solution of (2.1.1) *and the minus sign is taken when u is a weak solution of the adjoint* (2.1.2)*. Here V_r^+ is the usual (forward in time) corkscrew point of $\Psi_r(Y,s)$ and V_r^- is the backwards in time corkscrew point of $\Psi_r(Y,s)$.*

The final basic lemma that is reviewed here is the boundary Hölder continuity. We state the version from [HL01] in the case $B \equiv 0$.

Lemma 2.2.9 (Boundary Hölder continuity)**.** *Let Ω be a* $\mathrm{Lip}(1,1/2)$ *cylinder, let A satisfy* (1.0.2) *and $B \equiv 0$, let u be a weak solution of* (2.1.1) *or* (2.1.2) *in $T(\Delta_{2r}(Y,s))$. If $r > 0$ and u vanishes continuously on $\Delta_{2r}(Y,s)$ then there exists $0 < \alpha < 1$ such that for $(X,t) \in T(\Delta_{r/2}(Y,s))$*

$$u(X,t) \lesssim (\delta(X,t)/r)^\alpha \sup_{T(\Delta_r(Y,s))} u.$$

If $u \geq 0$ in $T(\Delta_{2r}(Y,s))$ then for $(X,t) \in T(\Delta_{r/2}(Y,s))$

$$u(X,t) \lesssim (\delta(X,t)/r)^\alpha u(V_r^{\pm}),$$

where the plus sign is taken when u is a weak solution of (2.1.1) *and the minus sign is taken when u is a weak solution of* (2.1.2)*.*

2.3 The Continuous Dirichlet Problem

In this section we review some of the classical results about the solvability of the continuous parabolic Dirichlet problem, we briefly define the parabolic measure, a consequence of this; and we introduce some properties of the Green's function and the parabolic measure.

2.3.1 The Parabolic Measure, Part I

It is a classical result via the Perron-Wiener-Brelot method [Ekl75; Ekl79] that the parabolic PDE (2.1.1) with continuous boundary data that decays to 0 as $t \to \pm\infty$ is uniquely solvable. We call this subclass of the continuous functions $C_0(\partial\Omega)$. The existence result uses lemmas 2.2.3, 2.2.5 and 2.2.9 and uniqueness follows from the maximum principle.

Theorem 2.3.1. *Let $f \in C_0$ then there exists a unique $u \in W^{1,2}_{\mathrm{loc}}(\Omega) \cap C_0(\overline{\Omega})$ such that u solves* (2.1.1) *weakly in the sense of definition 2.1.1 and $u|_{\partial\Omega} = f$.*

Definition 2.3.2 (Parabolic measure)**.** *Due to the Riesz representation theorem, for every point $(X,t) \in \Omega$ there exists a unique Borel measure $\omega^{(X,t)}$, called the parabolic measure, such that*

$$u(X,t) = \int_{\partial\Omega} f(Y,s)\,\mathrm{d}\omega^{(X,t)}(Y,s) \tag{2.3.1}$$

for all $f \in C_0(\partial\Omega)$.

Using the maximum principle (lemma 2.2.5) and that the class $C_0(\partial\Omega)$ is dense in all $L^p(\partial\Omega, \mathrm{d}\sigma)$ for $1 < p < \infty$ we can extend the solution operator from $C_0(\partial\Omega)$ to L^p. This gives us an interior solution. One then needs to check that the solution operator obeys the correct boundary estimate, in a L^p sense, so that $u(X,t) \to f(Y,s)$ as $(X,t) \to (Y,s)$ in a reasonable way. In essence this is the goal of chapters 3 and 4. See theorem 2.4.9 later for an exact statement. Furthermore, as $\omega^{(X,t)}$ is a Borel measure, we can use (2.3.1) to extend the solvability of (2.1.1) to classes of bounded Borel measurable functions f.

We write $\omega^{*(X,t)}$ to denote the parabolic measure of the adjoint equation, (2.1.2).

Remark 2.3.3. One alternative way to look at the parabolic measure is from a probabilistic viewpoint. If for simplicity we look at the heat equation $u_t = \Delta u$ then for a set $A \subset \partial\Omega$

$$\omega^{(X,t)}(A) = \left\{\begin{matrix}\text{the probability of Brownian motion starting at } (X,t) \\ \text{(backwards in time) and exiting } \Omega \text{ through } A\end{matrix}\right\}. \tag{2.3.2}$$

An illustration of this for a few realisations of Brownian motion can be seen in figure 2.1. Obviously this definition of the parabolic measure can be generalised to $u_t = \mathrm{div}(A\nabla u) + B \cdot \nabla u$ by considering a different stochastic process, however we have to assume more conditions on A and B for there to exist such a stochastic representation of our PDE.

Although we won't use this definition of the parabolic measure it does help to improve our intuition of the parabolic measure and the nature of the equation. For instance, one can see from this why our corkscrew points need the 'time-lag' — even though our equation has infinite speed of propagation; this lag ensures that the 'heat' has enough time to propagate through the 'material'. From the probabilistic viewpoint this is due to Brownian motion having normally distributed increments.

A question that we want to ask: is $\mathrm{d}\omega^{(X,t)}$ absolutely continuous with respect to the surface measure $\mathrm{d}\sigma$ for a general PDE of the form (2.1.1) in $\mathrm{Lip}(1, 1/2)$ cylinders? Kaufman and Wu [KW88] answered this question negatively by showing that one can construct a domain in $\mathbb{R} \times \mathbb{R}$ such that even for the heat equation, the nicest parabolic equation, these measures have differing Hausdorff dimension. See section 4.2.1 later for details. If the measures are absolutely

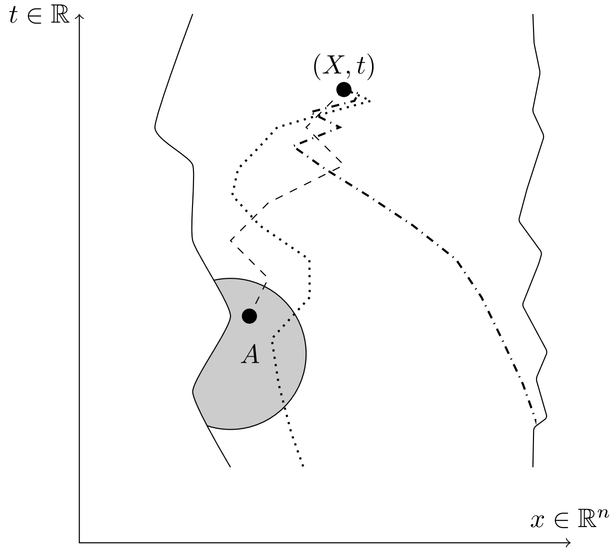

Figure 2.1: Three realisations of Brownian motion inside Ω illustrating how the parabolic measure of a set $A \subset \partial\Omega$, $\omega^{(X,t)}(A)$, can be represented in a probabilistic framework, c.f. remark 2.3.3.

continuous then the Radon-Nikodym derivative $\frac{\mathrm{d}\omega}{\mathrm{d}\sigma}$ belonging to a certain reverse Hölder class is equivalent to the solvability of the L^p Dirichlet problem, c.f. theorem 2.4.29 later.

Under the assumptions of definition 2.1.3 ($\mathrm{Lip}(1,1/2)$ domains) the parabolic measure is doubling [Nys97, Lemma 3.2]. The other statements in lemma 2.3.4 below are proved in [Nys97, p. 207 and Lemma 3.5].

Lemma 2.3.4 (Parabolic doubling, corkscrew point, see [Nys97] for more general statements in time-varying domains)**.** *Let Ω be a* $\mathrm{Lip}(1,1/2)$ *cylinder from definition 2.1.3 with character (ℓ,N,d). Let $\Delta_{2r} \subset \Delta_d$ be surface balls, and V_{2r} and V_d be their corkscrew points. Let A satisfy* (1.0.2)*, $B \equiv 0$ and ω^{V_d} be the parabolic measure of* (2.1.1)*. Then there exists $C = C(\lambda,\Lambda,n,\ell)$ such that the following properties hold:*

(i) $\omega^{V_d}(\Delta_d) \geq C$.

(ii) $\omega^{V_d}(\Delta_{2r}) \leq C\omega^{V_d}(\Delta_r)$. *(doubling)*

(iii) If $E \subset \Delta_{2r}$ is a Borel set then

$$\omega^{V_{2r}}(E) \sim \frac{\omega^{V_d}(E)}{\omega^{V_d}(\Delta_{2r})}. \tag{2.3.3}$$

The next lemma shows that the parabolic measure of different corkscrew points of large balls are comparable.

Lemma 2.3.5 (Change of corkscrew point)**.** *Let Ω be a Lip$(1,1/2)$ cylinder with character (ℓ,N,d). Let $\Delta_r(Y,s)$ be a surface ball with $r \sim d$, $(Y,s) \in \partial\Omega$, and let V_r and V_r' be two corkscrew points of $\Delta_r(Y,s)$ both later in time than $s+(2r)^2$. Let ω^{V_r} be the parabolic measure of* (2.1.1)*, A satisfy* (1.0.2)*, $B \equiv 0$ and if $E \subset \Delta_r(Y,s)$ is a Borel set then*

$$\omega^{V_r}(E) \sim \omega^{V_r'}(E). \tag{2.3.4}$$

The same result holds for the adjoint parabolic measure ω^{*V_r}, and when V_r and V_r' are corkscrew points earlier in time than $s-(2r)^2$.

Proof. The idea of this proof is to view $\omega^{V_r}(E)$ as $u(V_r)$, where u is the solution of (2.1.1) with boundary data χ_E (χ is the usual indicator function). We then set up to apply the maximum principle to an appropriately chosen domain $\partial\Omega \cap \{t \geq s'\}$. The domain, balls and points are illustrated in figure 2.2 on p. 14.

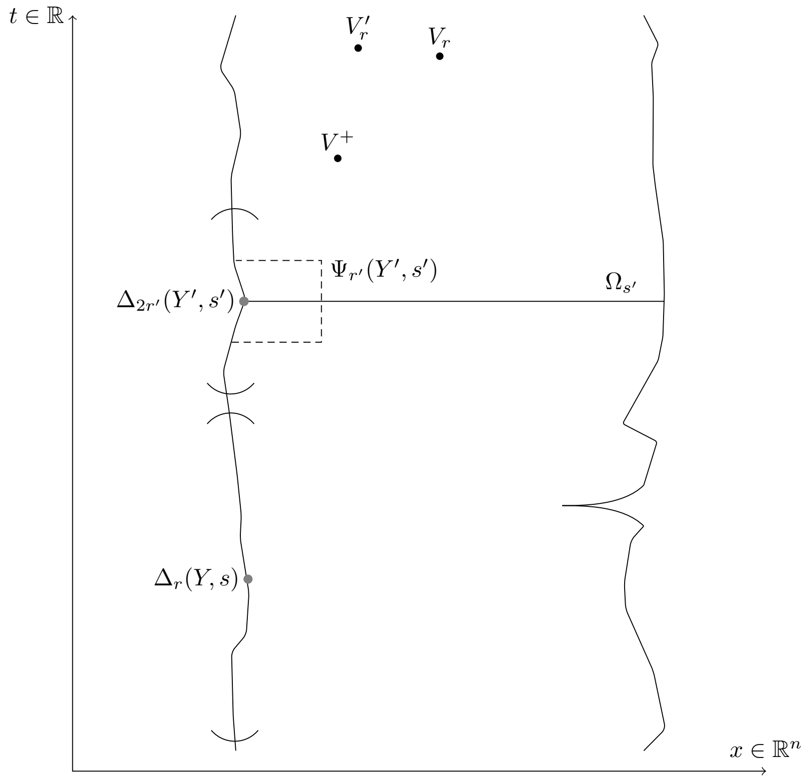

Figure 2.2: The situation in the proof of the change of corkscrew point lemma, lemma 2.3.5, on p. 13.

Let $(Y', s') \in \partial\Omega$ and r' be such that $\Delta_{2r'}(Y', s')$ is a surface ball later in time than $\Delta_r(Y, s)$, $E \subset \Delta_r(Y, s)$ and $\Delta_{2r'}(Y', s')$ are disjoint. Let V^+ be a corkscrew point of $\Psi_{2r'}(Y', s')$ earlier in time than V_r. Therefore the boundary data is 0 on $\Delta_{2r'}(Y', s')$ and we can apply lemma 2.2.8 to control u in $\Psi_{r'}(Y', s')$ by $u(V^+)$. In turn $u(V^+)$ is controlled by $u(V_r)$ using the Harnack inequality, lemma 2.2.4.

Since $r \sim \operatorname{diam} \Omega_{s'}$ by using Harnack chains, the Harnack inequality and varying Y' we can uniformly control u at the time s' by $u(V_r)$; that is we have $u(X, s') \lesssim u(V_r)$ for all $(X, s') \in \Omega_{s'}$. It follows by the maximum principle (lemma 2.2.5) applied to the domain $\partial\Omega \cap \{t \geq s'\}$ that $u(X, t) \lesssim u(V_r)$ for all $(X, t) \in \Omega \cap \{t \geq s'\}$. In particular, $u(V_r') \lesssim u(V_r)$ and therefore $\omega^{V_r}(E) \lesssim \omega^{V_r'}(E)$. Exchanging the roles of V_r and V_r' gives the reverse inequality. $\square$

Green's Function

We use the following properties of the Green's function in chapter 3. The existence of the Green's functions G and G^* in Ω for (2.1.1) and (2.1.2) respectively is well known. It follows from the Hölder continuity of the solution and a Perron-Wiener-Brelot style argument.

Lemma 2.3.6 ([Aro63; Fri64], see also [HL01])**.** *Let Ω be a Lip$(1, 1/2)$ cylinder, A satisfy* (1.0.2) *and $B = 0$ then the Green's function $G : \Omega \times \Omega \to \mathbb{R}$ for* (2.1.1) *has the following properties:*

(i) For all $\phi \in C_0^\infty(\mathbb{R}^{n+1})$

$$\phi(X, t) = \int_\Omega A\nabla\phi \cdot \nabla_Y G(X, t, \cdot) + G(X, t, \cdot)\phi_s \,\mathrm{d}Y \,\mathrm{d}s + \int_{\partial\Omega} \phi(Y, s) \,\mathrm{d}\omega^{(X,t)}(Y, s) \quad (2.3.5)$$

and

$$\phi(Y, s) = \int_\Omega A^*\nabla\phi \cdot \nabla_X G(\cdot, Y, s) - G(\cdot, Y, s)\phi_s \,\mathrm{d}X \,\mathrm{d}t + \int_{\partial\Omega} \phi(X, t) \,\mathrm{d}\omega^{*(Y,s)}(X, t). \quad (2.3.6)$$

(ii) $G(X, t, Y, s) = 0$ *for* $s > t$, (X, t), $(Y, s) \in \Omega$.

(iii) *For fixed* $(Y, s) \in \Omega$, $G(\cdot, Y, s)$ *is a solution to* (2.1.1) *in* $U \setminus \{(Y, s)\}$.

(iv) *For fixed* $(X, t) \in \Omega$, $G(X, t, \cdot)$ *is a solution to* (2.1.2)*, the adjoint equation, in* $\Omega \setminus \{(X, t)\}$.

(v) *If* (X, t), $(Y, s) \in \Omega$ *then* $G(X, t, \cdot)$ *and* $G(\cdot, Y, s)$ *extend continuously to* $\overline{\Omega}$ *provided both functions are defined to be zero on* $\partial\Omega$.

(vi) *The Green's function to the adjoint equation* G^* *is* $G^*(Y, s, X, t) = G(X, t, Y, s)$.

The representation formula, property (i) above, is due to the Green's function acting like a delta function when placed in the bilinear form in (2.1.3). We give a brief outline of (2.3.5) where $B = 0$.

Outline of Proof. Let

$$\mathcal{L}(u, v) = \int_\Omega (-uv_t + A\nabla u \cdot \nabla v) \,\mathrm{d}X \,\mathrm{d}t \tag{2.3.7}$$

then for all $\phi \in C_0^\infty(\Omega)$ $\mathcal{L}(G(\cdot, Y, s), \phi) = \phi(Y, s)$. Hence $L(G(X, t, Y, s)) = -\delta_{(X,t)}(Y, s)$ in the sense of distributions, where L is the parabolic operator $L = \operatorname{div}(A\nabla) - \partial_t$ acting in X and t. Let $\psi \in W_0^{1,2}(\Omega)$ then

$$G(\psi)(X, t) = \int_\Omega G(X, t, Y, s)\psi(Y, s) \,\mathrm{d}Y \,\mathrm{d}s$$

and $LG(\psi) = \psi$ [Aro63]. Therefore if $\phi \in C_0^\infty(\mathbb{R}^{n+1})$ let u be the solution to (2.1.1) given by (2.3.1) with boundary data ϕ, that is

$$u(X, t) = \int_{\partial\Omega} \phi(Y, s) \,\mathrm{d}\omega^{(X,t)}(Y, s).$$

Then $\phi - u \in W_0^{1,2}(\Omega)$, $L(\phi - u) = L\phi$ and hence $G(L\phi) = \phi - u$ via the adjoint operators. Integrating this by parts gives property (i) above:

$$\phi(X, t) = \int_\Omega A\nabla\phi \cdot \nabla_Y G(X, t, \cdot) + G(X, t, \cdot)\phi_s \,\mathrm{d}Y \,\mathrm{d}s + \int_{\partial\Omega} \phi(Y, s) \,\mathrm{d}\omega^{(X,t)}(Y, s). \qquad \square$$

The following lemma is a consequence of [Nys97]. We state it for the adjoint equation (2.1.2) in Ω as we apply the lemma in this context in chapter 3. This lemma was originally stated in Lipschitz cylinders in [FGS86, Theorem 1.4; FS97, Theorem 4] and was extended to Lip$(1, 1/2)$ cylinders in [Nys97].

Lemma 2.3.7. *Let* Ω *be a Lip*$(1, 1/2)$ *cylinder, A satisfy* (1.0.2) *and* $B = 0$*. Let* G^* *be the Green's function and* ω^* *be the parabolic measure associated to* (2.1.2)*. Let* $\Delta_r \subset \Delta_d$ *be the surface balls on* $\partial\Omega$ *such that* $\Delta_{2r} \subset \Delta_d$*. Then there exists constants depending on* n*,* λ *and* Λ *and character of the domain* Ω *such that*

$$r^n G^*(V^-(\Delta_d), V^-(\Delta_r)) \sim \omega^{*V^-(\Delta_d)}(\Delta_r). \tag{2.3.8}$$

Here $V^-(\Delta_r)$ *and* $V^-(\Delta_d)$ *are backward in time corkscrew points of the parabolic surface balls* Δ_r *and* Δ_d *respectively, as in definition 2.2.6.*

2.4 The L^p Dirichlet Problem

We are investigating the solvability of (2.1.1) with the weakest possible boundary data. If our data lives in $L^p(\partial\Omega)$ then we expect our solution $u \in L^p_{1/p}(\Omega)$ — that is roughly: $u \in L^p$ and $1/p$ of a derivative of u lives in L^p. However, this is exactly the borderline case where we do not have a trace theorem for this function space [JW84; JK95].

Therefore we introduce the non-tangential maximal function which serves as a replacement for the trace theorem and allows us to assign a boundary value to a solution u. We also present square functions and define the L^p regularity and Dirichlet problems. In section 2.4.2 we define the kernel function and in section 2.4.3 we study the theory of A_p, B_p and A_∞ weights. Section 2.4.4 then uses this theory of weights and the kernel function to deduce the solvability of the L^p Dirichlet problem and look at perturbation results.

2.4.1 Parabolic Non-tangential Cones, Non-tangential Maximal Functions and the Square Function

We now define parabolic non-tangential cones, non-tangential maximal functions and square functions. For simplicity of the definitions we state these definitions in a (local) coordinate system where Ω is given by just one graph: $\Omega = \{(x_0, x, t) : x_0 > \phi(x,t)\}$. If we change the choice of coordinates then this leads to a different set of cones; however this does not make a difference since it only changes constants in the estimates of the non-tangential maximal functions and square functions. By a geometric argument the norms defined using different sets of cones are equivalent.

For a constant $a > 0$ (the aperture), we define the *parabolic non-tangential cone* at a point $(x_0, x, t) \in \partial\Omega$ as

$$\Gamma_a(x_0, x, t) = \left\{(y_0, y, s) \in \Omega : |y - x| + |s - t|^{1/2} < a(y_0 - x_0),\ x_0 < y_0\right\}.$$

We define the truncated cone Γ at the height h as

$$\Gamma_a^h(x_0, x, t) = \left\{(y_0, y, s) \in \Omega : |y - x| + |s - t|^{1/2} < a(y_0 - x_0),\ x_0 < y_0 < x_0 + h\right\}.$$

To ensure the cones are still contained in Ω we need the aperture $a \leq \ell$ — the $\mathrm{Lip}(1, 1/2)$ constant of the domain Ω.

Definition 2.4.1 (Non-tangential maximal function). *For a function $u : \Omega \to \mathbb{R}$ we define the non-tangential maximal function $N_a(u) : \partial\Omega \to \mathbb{R}$ and its truncated version at a height h to be*

$$\begin{aligned} N_a(u)(X,t) &= \sup_{(Y,s)\in\Gamma_a(X,t)} |u(Y,s)|, \\ N_a^h(u)(X,t) &= \sup_{(Y,s)\in\Gamma_a^h(X,t)} |u(Y,s)|. \end{aligned} \tag{2.4.1}$$

We also define the following L^p variant of the non-tangential maximal function, which is used in the regularity problem. Here we can't use the usual non-tangential maximal function since a priori $\nabla u \in L^2$ and $\nabla u \notin L^\infty$.

$$\widetilde{N}_{a,p}^r(u)(X,t) = \sup_{(Y,s)\in\Gamma_a^h(X,t)} \left(\fint_{B_{\delta(Y,s)/2}(Y,s)} |u(Z,\tau)|^p \, \mathrm{d}Z \, \mathrm{d}\tau \right)^{1/p}. \tag{2.4.2}$$

Remark 2.4.2. It can be shown using lemma 2.2.1 if u is a solution to (2.1.1) then

$$\|\widetilde{N}_2(u)\|_{L^p(\partial\Omega,\, \mathrm{d}\sigma)} \sim \|N(u)\|_{L^p(\partial\Omega,\, \mathrm{d}\sigma)}.$$

The square function (also historically called the Lusin area integral) is a classical maximal operator that is intimately related to the non-tangential maximal function, the space BMO, Carleson measures and many other areas of harmonic analysis [Ste93]. We use it to control the spatial derivatives of u. When the square function and non-tangential maximal function are comparable this implies solvability of the L^p Dirichlet problem for all $p > p'$, for a potentially large p', (or as we see later in section 2.4.4 an A_∞ result) [Dah80; KKPT00; Riv03]; see theorem 4.1.2 for the exact statement.

Definition 2.4.3 (Square function). *For a function $u : \Omega \to \mathbb{R}$, the square function $S_a(u)$:*

$\partial\Omega \to \mathbb{R}$ and its truncated version at a height r are defined as

$$S_a(u)(Y,s) = \left(\int_{\Gamma_a(Y,s)} |\nabla u(X,t)|^2 \delta(X,t)^{-n} \, \mathrm{d}X \, \mathrm{d}t \right)^{1/p}, \\ S_a^r(u)(Y,s) = \left(\int_{\Gamma_a^r(Y,s)} |\nabla u(X,t)|^2 \delta(X,t)^{-n} \, \mathrm{d}X \, \mathrm{d}t \right)^{1/p}. \tag{2.4.3}$$

By applying Fubini we also have in the upper half space $U = \{(x_0, x, t) : x_0 > 0\}$

$$\|S_a(u)\|^p_{L^p(\partial U)} \sim \int_U |\nabla u|^2 x_0 \, \mathrm{d}x_0 \, \mathrm{d}x \, \mathrm{d}t. \tag{2.4.4}$$

We often suppress the parameters h and a when they are not needed. The hiding of the aperture a is always justified since the non-tangential maximal functions and square functions defined using two different sets of cones are equivalent. The following lemma is an easy adaptation of [DH18, Lemma 4.2], where it is proved for admissible parabolic domains and non-tangential maximal functions (c.f. definition 4.2.20). The parabolic version was in turn adapted from the elliptic result [Din02, Lemma 2.3].

Lemma 2.4.4. *Let $h > 0$ and $0 < a < b \leq \ell/2$, where ℓ is the $\mathrm{Lip}(1, 1/2)$ norm of Ω. Consider the non-tangential maximal functions defined using two sets of cones Γ_a^h and Γ_b^h. Then for any $p > 0$ there exists a constant $C_p > 0$ such that for all $u : \Omega \to \mathbb{R}$*

$$N_a^h(u) \leq N_b^h(u) \quad \text{and} \quad \|N_b^h(u)\|_{L^p(\partial\Omega)} \leq C_p \|N_a^h(u)\|_{L^p(\partial\Omega)}.$$

Furthermore, the same result holds for square functions.

The proof below is based upon the following covering lemma from [CT75, Lemma 1.6].

Lemma 2.4.5. *Let $E \subset \mathbb{R}^{n-1} \times \mathbb{R}$ and suppose that for each $(x, t) \in E$ there exists a ball $\Delta_{r(x,t)}(x, t)$ centred at (x, t). If the side lengths $r(x, t)$ are bounded in E then there exists a disjoint countable family $\Delta_{r_j(x_j,t_j)}(x_j, t_j)$ of these balls such that the following properties hold:*

(i) $E \subset \bigcup_j \Delta_{3r_j(x_j,t_j)}(x_j, t_j)$.

(ii) For all $(x, t) \in E$ there exists (x_j, t_j) such that $\Delta_{r(x,t)}(x, t) \subset \Delta_{5r_j(x_j,t_j)}(x_j, t_j)$.

Proof of lemma 2.4.4. We only prove the lemma for non-tangential maximal functions but the same proof holds for square functions. Since $b < \ell/2$ after taking a pullback mapping we may assume that we're in the setting of the upper half space U. The inequality $N_a^h(u) \leq N_b^h(u)$ is obvious since $\Gamma_a \subset \Gamma_b$ for $a < b$.

The proof of the opposite direction is based upon [BG72, Lemma 2; Ken80, Lemma 2.3] using distribution functions. We wish to show for any fixed $\lambda > 0$

$$|\{(x, t) \in \partial U : N_b^r(u)(x, t) > \lambda\}| \lesssim |\{(x, t) \in \partial U : N_a^r(u)(x, t) > \lambda\}|.$$

The conclusion then follows immediately from the distribution functions. If we set $E_b(\lambda) = \{(x, t) \in \partial U : N_b^h(u)(x, t) > \lambda\}$ then $\int_{\partial U} N_b^h(u)^p = p \int_0^\infty |E_b(\lambda)| \lambda^{p-1} \, \mathrm{d}\lambda$ and the same is true for E_a.

The proof is now based on two geometric observations. First if $(z_0, z, \tau) \in \Gamma_b^h(x, t)$ then $(x, t) \in \Delta_{bz_0}(z, \tau)$, and second if $(y, s) \in \Delta_{ax_0/n}(x, t)$ and $0 < x_0 < h$ then $(x_0, x, t) \in \Gamma_a^h(y, s)$. Let $(x, t) \in E_b(\lambda)$ then there exists a point $(z_0, z, \tau) \in \Gamma_b^h(x, t)$ such that $|u(z_0, z, \tau)| > \lambda$. Using the first observation $(x, t) \in \Delta_{bz_0}(z, \tau)$. If $(y, s) \in \Delta_{az_0/n}(z, \tau)$ then the second observation gives $(z_0, z, \tau) \in \Gamma_a^h(y, s)$ and hence $N_a^h(y, s) > \lambda$. Consequently $\Delta_{az_0/n}(z, \tau) \subset E_a(\lambda)$.

Let $r(x, t) > 0$ be the smallest number such that $\Delta_{az_0/n}(z, \tau) \subset \Delta_{r(x,t)}(x, t)$. Another simple geometric consideration of cones shows the existence of $C(a, b) > 0$ such that $\left|\Delta_{r(x,t)}(x, t)\right| \leq C \left|\Delta_{az_0/n}(z, \tau)\right|$. We may assume $\sup_{(x,t) \in E_b(\lambda)} r(x, t)$ is finite otherwise $E_b(\lambda)$ and $E_a(\lambda)$ both contain balls of arbitrarily large radius and hence the claim is trivial. Applying property (i) of lemma 2.4.5 we may extract a countable disjoint family of balls $\Delta_{r_j(x_j,t_j)}(x_j,t_j)$ such that

$$\begin{aligned}|E_b(\lambda)| &\leq \sum_j |\Delta_{3r_j(x_j,t_j)}(x_j,t_j)| \lesssim \sum_j |\Delta_{r_j(x_j,t_j)}(x_j,t_j)| \\ &\lesssim C\sum_j |\Delta_{a/nz_{0i}}(z_j,\tau_j)| \leq C|E_a(\lambda)|,\end{aligned}$$

where we have used the fact that $\Delta_{a/nz_{0i}}(z_j,\tau_j)$ are disjoint and contained in $E_a(\lambda)$. □

We are now in the position to define the L^p Dirichlet and the L^p regularity problems. The solvability of the L^p Dirichlet problem is equivalent to a certain type of mutual absolute continuity (A_∞ or B_p result) between the surface measure and the parabolic measure — see theorem 2.4.29 for an exact statement. This is the motivation for studying weights in section 2.4.3 and explains why the counter-example in [KW88] is such a strong result.

Definition 2.4.6 (L^p Dirichlet problem)**.** *We say the L^p Dirichlet problem for the equation* (2.1.1) *with boundary data in $L^p(\partial\Omega, \mathrm{d}\sigma)$ is solvable if the unique solution u in Ω for any boundary data $f \in C_0(\partial\Omega)\cap L^p(\partial\Omega, \mathrm{d}\sigma)$ satisfies $u|_{\partial\Omega} = f$ a.e. and the following non-tangential maximal function estimate holds*

$$\|N(u)\|_{L^p(\partial\Omega,\,\mathrm{d}\sigma)} \lesssim \|f\|_{L^p(\partial\Omega,\,\mathrm{d}\sigma)}, \tag{2.4.5}$$

with the implicit constant depending only on the ellipticity constants, n, p and triple (ℓ, N, d) of the domain. When (2.4.5) *holds we say that the equation* (2.1.1) *has the property $(D)_p$ in Ω. The property $(D^*)_{p'}$ for the adjoint equation* (2.1.2) *is defined analogously.*

The L^p regularity problem assumes that we have knowledge (in the L^p sense) of a derivative of the boundary data f. In the setting of parabolic equations this derivative is the parabolic derivative, which in the L^p case is examined in detail in section 3.2. Roughly speaking the space $L^p_{1,1/2}$ consists of functions f such that $f \in L^p$, $\nabla f \in L^p$ and half a derivative in time belongs to L^p — denoted by $D^t_{1/2}f \in L^p$.

Definition 2.4.7 (L^p regularity problem)**.** *Following [Bro89b; Bro90; HL96; HL99] we say the L^p Regularity problem for the equation* (2.1.1) *is solvable if the unique solution u in Ω with boundary data $f \in C_0(\partial\Omega) \cap L^p_{1,1/2}(\partial\Omega, \mathrm{d}\sigma)$ satisfies $u|_{\partial\Omega} = f$ a.e. and the following non-tangential maximal function estimate holds*

$$\|\widetilde{N}_2(\nabla u)\|_{L^p(\partial\Omega,\,\mathrm{d}\sigma)} \lesssim \|f\|_{L^p_{1,1/2}(\partial\Omega,\,\mathrm{d}\sigma)}, \tag{2.4.6}$$

with the implied constants depending only on the ellipticity constants, n, p and triple (ℓ, N, d) of the domain. Here $\widetilde{N}_2$ denotes the L^2 based non-tangential maximal function from (2.4.2)*. When* (2.4.6) *holds we say that* (2.1.1) *has the property $(R)_p$ in Ω.*

Here the use of the L^2 based non-tangential maximal function is natural since $\nabla u \in L^2_{\mathrm{loc}}(\Omega)$. In general, better smoothness of the gradient cannot be expected unless we assume more smoothness of the coefficients of the parabolic operator.

Remark 2.4.8. Some authors [Bro87; Mit01; Nys06; CRS15] also require $\|\widetilde{N}_2(D^t_{1/2}u)\|_{L^p} \lesssim \|f\|_{L^p_{1,1/2}}$ or $\|\widetilde{N}_2(HD^t_{1/2}u)\|_{L^p} \lesssim \|f\|_{L^p_{1,1/2}}$, where H is the Hilbert transform in the time variable. For our result in chapter 3 we do not need to assume this, hence our notion of solvability is slightly weaker than that of the authors above. It follows therefore that the $(R)_p$ solvability in the sense of [Bro87; Mit01; Nys06; CRS15] implies solvability in the sense of definition 2.4.7.

The reason why we only have to test the non-tangential maximal function against continuous L^p functions is due to the following theorem.

Theorem 2.4.9. *If Ω be a* Lip(1, 1/2) *cylinder and the property (D_p) holds, then for all $f \in L^p(\partial\Omega, \mathrm{d}\sigma)$ there exists a unique solution u to* (2.1.1)*, as in definition 2.1.1, such that $u \to f$ non-tangentially a.e. and $\|Nu\|_{L^p(\partial\Omega)} \lesssim \|f\|_{L^p(\partial\Omega)}$.*

The proof of this theorem is standard and goes via a density argument using theorem 2.3.1. The existence is usually proved as in [Ken94, Theorem 1.7.7] and the uniqueness is proved in [Nys97, Theorem 6.3] using the Green's function representation formula, property (i) of lemma 2.3.6.

2.4.2 The Kernel Function

An important tool in studying the Dirichlet problem for $f \in L^p(\mathrm{d}\omega)$ in the parabolic and elliptic setting is the kernel function, which was developed in [Kem72; FGS86; Nys97]. Instead we are studying the L^p Dirichlet problem with respect to the surface measure $\mathrm{d}\sigma$ but we do still use the kernel function in remark 2.4.28.

Let (X_0, T) be a fixed reference point in Ω and Q_r be a parabolic boundary cube.

Definition 2.4.10. *For Ω a* Lip$(1, 1/2)$ *cylinder and $(X, t) \in \Omega$, the unique kernel function $K : \partial\Omega \to \mathbb{R} \cup \{\infty\}$, normalised at (X_0, T), is defined as*

$$K^{(X,t)}(Y, s) = \lim_{r \to 0+} \frac{\omega^{(X,t)}(Q_r(Y, s))}{\omega^{(X_0,T)}(Q_r(Y, s))}, \tag{2.4.7}$$

where ω is the parabolic measure from definition 2.3.2.

Equivalently one may define the kernel function as the unique function which satisfies the following conditions for each $(Y, s) \in \partial\Omega$:

(1) $K^{(X,t)}(Y, s) \geq 0$ for each $(X, t) \in \Omega$ and $K^{(X_0,T)}(Y, s) = 1$.

(2) $K^{(\cdot)}(Y, s)$ is a solution to (2.1.1) *in Ω.*

(3) $K^{(\cdot)}(Y, s) \in C(\overline{\Omega} \setminus \{(Y, s)\})$ and

$$\lim_{(X,t) \to (Z,\tau)} K^{(X,t)}(Y, s) = 0$$

if $(Z, \tau) \in C(\overline{\Omega} \setminus \{(Y, s)\})$.

Finally if $s \geq T$ then we extend K by defining it to be 0.

Definition 2.4.11. *The Hardy-Littlewood maximal function of a measure μ with respect to $\omega^{(X_0,T)}$ is*

$$M_{\omega^{(X_0,T)}}(\mu)(Z, \tau) = \sup_{r>0} \frac{|\mu|(\Delta_r(Y, s))}{\omega^{(X_0,T)}(\Delta_r(Y, s))}, \tag{2.4.8}$$

where μ is a measure on $\partial\Omega$ and $(Y, s) \in \partial\Omega$.

Theorem 2.4.12 ([Nys97, Theorem 4.3])**.** *Let μ be a finite, signed Borel measure on $\partial\Omega$, $t < \tau$ and let*

$$u(X, t) = \int_{\partial\Omega} K^{(X,t)}(Y, s)\, \mathrm{d}\mu(Y, s)$$

then $N(u)(Z, \tau) \lesssim M_{\omega^{(X_0,T)}}(\mu)(Z, \tau)$ for $(Z, \tau) \in \partial\Omega$. Furthermore, if $\mu \geq 0$ then $M_{\omega^{(X_0,T)}}(\mu)(Z, \tau) \lesssim N(u)(Z, \tau)$.

Definition 2.4.13. *A region $\Omega \subset \mathbb{R}^{n+1}$ is parabolically star shaped with respect to the vertex (X_0, T) if for each $(Y, s) \in \partial\Omega$ there exists a finite parabolic ray with vertex (X_0, T) and endpoint (Y, s) which is contained in Ω.*

Theorem 2.4.14 ([FGS86, Theorem 2.10; Nys97, Theorem 4.4])**.** *Let Ω be a parabolically star shaped* Lip$(1, 1/2)$ *cylinder with respect to (X_0, T). If u solves* (2.1.1) *and $u \geq 0$ then there exists a unique Borel measure μ on $\partial\Omega$ such that if $(X, t) \in \Omega$ with $t \leq T$ then*

$$u(X, t) = \int_{\partial\Omega} K^{(X,t)}(Y, s)\, \mathrm{d}\mu(Y, s), \tag{2.4.9}$$

where K is the kernel function for (2.1.1) *normalised at $(X_0, T + 1)$.*

2.4.3 The Theory of Weights

The solvability of the L^p Dirichlet problem is intimately intertwined with the parabolic measure and the theory of weights. Therefore we give an overview of this theory. It was initially developed in the setting of elliptic PDE in [FKP91], with some of the first results in [Dah77; Dah79]. We start by introducing the classical theory of Muckenhoupt weights [Muc72] on $\mathbb{R}^n$ with the Lebesgue measure and then later apply it to our $\mathrm{Lip}(1, 1/2)$ cylinders in section 2.4.4. The classes of weights that we define here can just as easily be defined with respect to a different measure, for instance our σ from definition 2.1.7.

Definition 2.4.15. *A weight is a function $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that $w \geq 0$ a.e. This means that if w is a weight then $1/w$ is also a weight. Given a weight w and a measurable set E we define the w-measure of a set to be $w(E) = \int_E w(x)\,\mathrm{d}x$.*

We can naturally define weighted Lebesgue spaces $L^p(w)$ and weak weighted Lebesgue spaces. Throughout this discussion on weights we use the definition of the Hardy-Littlewood maximal function with the supremum taken over all cubes Q containing x,

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|\,\mathrm{d}y.$$

It is a standard harmonic analysis result that M is strong (p,p) for $1 < p < \infty$ and weak $(1,1)$ [Duo01]. To stop this section growing too large, we restrict ourselves to the case $p > 1$; however there are analogous results for the $p = 1$ endpoint, which are contained in the works cited. We wish to understand when M is bounded if $\mathrm{d}x$ is replaced by $w(x)\,\mathrm{d}x$. That is, we want to characterise the weights w such that the strong type (p,p) inequality holds

$$\int_{\mathbb{R}^n} Mf(x)^p w(x)\,\mathrm{d}x \leq C \int_{\mathbb{R}^n} |f(x)|^p w(x)\,\mathrm{d}x, \tag{2.4.10}$$

and characterise the weights w such that the weak type (p,p) inequality holds

$$w(\{x \in \mathbb{R}^n : Mf(x) > \lambda\}) \leq \frac{C}{\lambda^p} \int_{\mathbb{R}^n} |f(x)|^p w(x)\,\mathrm{d}x. \tag{2.4.11}$$

To establish this we first define a class of weights that satisfy these inequalities.

Definition 2.4.16. *For $1 < p < \infty$ a weight w is said to be of class A_p if*

$$\left(\frac{1}{|Q|}\int_Q w\right)\left(\frac{1}{|Q|}\int_Q w^{1-p'}\right)^{p-1} \leq C, \tag{2.4.12}$$

where C is independent of Q and the prime denotes the Hölder conjugate. Let $[w]_{A_p}$ denote the best constant C.

Example 2.4.17. The function $|x|^a \in A_p$ exactly when $-n < a < n(p-1)$ [Gra09, p. 286]. This allows us to construct an example of a doubling measure that is not in A_p — when $a > n(p-1)$ the measure $|x|^a\,\mathrm{d}x$ is still a doubling measure.

Proposition 2.4.18 (Properties of A_p weights [Gra09; Duo01])**.** *Let $1 < p < \infty$ then the following properties hold:*

(i) The classes A_p are increasing: if $p < q$ then $A_p \subset A_q$ and $[w]_{A_p} \leq [w]_{A_q}$.

(ii) $A_p = \cup_{q<p} A_q$ and the classes A_p are open: for any $w \in A_p$ there exists an $\varepsilon > 0$ such that $w \in A_{p+\varepsilon}$.

(iii) If $w \in A_p$ then there exists an $\varepsilon > 0$ such that $w^{1+\varepsilon} \in A_p$.

(iv) $w \in A_p$ if and only if $w^{1-p'} \in A_{p'}$, where p' is the Hölder conjugate of p. Therefore $w \in A_2$ if and only if $w^{-1} \in A_2$.

(v) The measure $w(x)\,\mathrm{d}x$ is a doubling measure.

(vi) $[w]_{A_p} \geq 1$ for all $w \in A_p$ with equality if and only if w is constant.

(vii) If $w \in A_p$ then there exists $\delta > 0$ such that given a cube Q and $E \subset Q$ then

$$\frac{w(E)}{w(Q)} \lesssim \left(\frac{|E|}{|Q|}\right)^{\delta}.$$

The next property of A_p weights is a scale invariant version of absolute continuity.

Lemma 2.4.19 ([Duo01, Lemma 7.5])**.** *Let $w \in A_p$ for some $1 < p < \infty$ and $0 < \alpha < 1$. Then there exists $0 < \beta < 1$ such that given a cube Q and $E \subset Q$ then*

$$\frac{|E|}{|Q|} < \alpha \implies \frac{w(E)}{w(Q)} < \beta.$$

Theorem 2.4.20 (Weak and strong type (p,p) inequality [Gra09; Duo01, Theorem 7.3])**.** *Let $1 < p < \infty$ then the weak type (p,p) inequality holds, $M : L^p(w) \to L^{p,\infty}$, if and only if $w \in A_p$. In addition, the strong type (p,p) inequality holds, M is bounded on $L^p(w)$, if and only if $w \in A_p$.*

A useful property of weights is that we can extrapolate L^p boundedness from A_p.

Theorem 2.4.21 (Rubio de Francia Extrapolation Theorem, [CMP11, Theorem 1.4])**.** *Let $1 \leq r < \infty$ and suppose that an operator T is bounded on $L^r(w)$ for all $w \in A_r$ with the operator norm depending only on $[w]_{A_r}$. Then T is bounded on $L^p(w)$, $1 < p < \infty$, for any $w \in A_p$ with the operator norm depending only on $[w]_{A_p}$.*

Definition 2.4.22. *We say that a weight w belongs to the reverse-Hölder class B_p if for all cubes Q w satisfies the following reverse Hölder inequality*

$$\left(\frac{1}{|Q|}\int_Q w^p\,\mathrm{d}x\right)^{1/p} \lesssim \frac{1}{|Q|}\int_Q w\,\mathrm{d}x. \tag{2.4.13}$$

Theorem 2.4.23 ([Muc72])**.** *Let $1 < p < \infty$ then $w \in B_p$ if and only if $w \in A_s$ for some $s > 1$. Furthermore, we have the precise relationship that $w \in B_p$ is equivalent to $w^{-1} \in A_{p'}$, where p and p' are Hölder conjugates.*

Many of the properties of the class of B_p weights can now be deduced from the properties of A_p in proposition 2.4.18. One important property to highlight is that the class B_p is open [Geh73]; if $w \in B_p$ then $w \in B_{p+\varepsilon}$ for some $\varepsilon > 0$.

The Class of A_∞ Weights

Sometimes we might not know which A_p or B_p class a weight belongs to but instead that it satisfies properties that are a common to all A_p or B_p weights (independent p). We have seen examples of such properties in proposition 2.4.18 and lemma 2.4.19. The following A_∞ class of weights can be intuitively thought of as weights (or measures) that satisfy a scale invariant version of mutual absolute continuity.

Definition 2.4.24. *We define A_∞ to be the union of all the A_p classes of weights,*

$$A_\infty = \bigcup_{1 \leq p < \infty} A_p. \tag{2.4.14}$$

We use the reverse Jensen inequality, seen in the next theorem, to define $[w]_{A_\infty}$,

$$[w]_{A_\infty} = \sup_{\text{cubes } Q} \left(\frac{1}{|Q|}\int_Q w\right) \exp\left(\frac{1}{|Q|}\int_Q \log(1/w)\right). \tag{2.4.15}$$

Theorem 2.4.25 (Characterisations of A_∞ weights [Muc74; CF74; GR85, §IV.2])**.** *Let w be a weight then w is in A_∞ if and only if one of the following conditions holds:*

(1) w is in A_p for some $1 < p < \infty$.

(2) w is in B_p for some $1 < p < \infty$; therefore $A_\infty = \bigcup_{1 \le p < \infty} B_p$.

(3) There exists $0 < \alpha, \beta < 1$ such that for all cubes Q and all measurable $E \subset Q$ we have

$$\frac{w(E)}{w(Q)} < \alpha \implies \frac{|E|}{|Q|} < \beta.$$

(4) There exists $0 < \alpha, \beta < 1$ such that for all cubes Q and all measurable $E \subset Q$ we have

$$\frac{|E|}{|Q|} < \alpha \implies \frac{w(E)}{w(Q)} < \beta.$$

(5) There exists $\delta > 0$ and C such that for all cubes Q and all measurable $E \subset Q$ we have

$$\frac{w(E)}{w(Q)} \le C \left(\frac{|E|}{|Q|} \right)^{\delta}.$$

(6) (Reverse Jensen inequality, [GR85, p. 405]) There exists C such that for all cubes Q

$$\left(\frac{1}{|Q|} \int_Q w \right) \exp \left(\frac{1}{|Q|} \int_Q \log(1/w) \right) \le C.$$

(7) There exists $0 < \gamma, \delta < 1$ such that for all cubes $Q \subset \mathbb{R}^n$ we have

$$\left| \left\{ x \in Q : w(x) \le \frac{\gamma}{|Q|} \int_Q w \right\} \right| \le \delta |Q|.$$

By conditions (3) and (4) a weight $w \in A_\infty$ is a scale invariant version of mutual absolute continuity. In addition, if we denote $A_\infty(\mathrm{d}x)$ as the A_∞ class defined above with respect to the Lebesgue measure then the A_∞ class is an equivalence class; a measure $\omega \in A_\infty(\mathrm{d}\mu) \iff \mu \in A_\infty(\mathrm{d}\omega)$.

Lemma 2.4.26 ([Gra09, p. 308])**.** *A weight w is in A_p if and only if w and $w^{\frac{-1}{1-p}}$ are in A_∞.*

2.4.4 The Parabolic Measure, Part II

We now define A_∞ and B_p on $\mathrm{Lip}(1, 1/2)$ cylinders with respect to σ, our surface measure from definition 2.1.7. Comparing these definitions to their elliptic versions in [FKP91] one can see that some care has been taken since the parabolic measures at different points are not mutually absolutely continuous to each other. This can be seen from the time irreversibility.

Definition 2.4.27 (A_∞ and B_p, [DPP17])**.** *Let Ω be a $\mathrm{Lip}(1, 1/2)$ cylinder from definition 2.1.3 with character (ℓ, N, d). Let V_d be the corkscrew point of Δ_d. We use condition (3) of theorem 2.4.25 to say that the parabolic measure ω^{V_d} is in $A_\infty(\mathrm{d}\sigma, \Delta_d)$ if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for any ball $\Delta \subset \Delta_d$ and measurable $E \subset \Delta$ we have*

$$\frac{\omega^{V_d}(E)}{\omega^{V_d}(\Delta)} < \delta \implies \frac{\sigma(E)}{\sigma(\Delta)} < \varepsilon. \tag{2.4.16}$$

The parabolic measure ω is in $A_\infty(\mathrm{d}\sigma)$ if ω^{V_d} belongs to $A_\infty(\mathrm{d}\sigma, \Delta_d)$ for all Δ_d. If A_∞ holds then ω^{V_d} and σ are mutually absolutely continuous and hence one can write $\mathrm{d}\omega^{V_d} = k^{V_d}\,\mathrm{d}\sigma$, where $k^{V_d} = \frac{\mathrm{d}\omega^{V_d}}{\mathrm{d}\sigma}$ is the Radon-Nikodym derivative.

For $1 < p < \infty$ we say that ω belongs to the reverse-Hölder class $B_p(\mathrm{d}\sigma)$ if for all Δ_d the kernel k^{V_d} satisfies the reverse Hölder inequality

$$\left(\frac{1}{\sigma(\Delta)} \int_\Delta \left(k^{V_d}\right)^p \,\mathrm{d}\sigma \right)^{1/p} \lesssim \frac{1}{\sigma(\Delta)} \int_\Delta k^{V_d}\,\mathrm{d}\sigma, \tag{2.4.17}$$

for all balls $\Delta \subset \Delta_d$.

Remark 2.4.28 ([Nys97, Remark 6.2])**.** Let $f \in C_0(\partial\Omega) \cap L^p(\partial\Omega, \mathrm{d}\sigma)$ and u be as in (2.3.1), that is $u(X,t) = \int_{\partial\Omega} f(Y,s)\,\mathrm{d}\omega^{(X,t)}(Y,s)$. For the moment we ignore the point (X,t) that the parabolic measure is taken with respect to. If $\omega \in B_{p'}(\mathrm{d}\sigma)$ and k is the Radon-Nikodym derivative of ω with respect to σ then by theorem 2.4.23 k^{-1} is an $A_p(\mathrm{d}\sigma)$ weight. Therefore by theorems 2.4.12 and 2.4.20 we have

$$\begin{aligned}\int_{\partial\Omega} |N(u)|^p \,\mathrm{d}\sigma &\lesssim \int_{\partial\Omega} |M_\omega(f)|^p \,\mathrm{d}\sigma \lesssim \int_{\partial\Omega} |M_\omega(f)|^p k^{-1} \,\mathrm{d}\omega \\ &\lesssim \int_{\partial\Omega} |f|^p k^{-1} \,\mathrm{d}\omega = \int_{\partial\Omega} |f|^p \,\mathrm{d}\sigma\end{aligned}$$

and hence $\|N(u)\|_{L^p(\partial\Omega)} \lesssim \|f\|_{L^p(\partial\Omega)}$ for $f \in C_0(\partial\Omega) \cap L^p(\partial\Omega, \mathrm{d}\sigma)$. This is exactly the (D_p) condition and this argument shows $\omega \in B_{p'}(\mathrm{d}\sigma)$ is equivalent to the (D_p) condition. This discussion can be made rigorous using the results of section 2.2 and a covering argument.

Following on from remark 2.4.28 we state the following well known theorem, which is essentially already proved by the preceding remark and theorems.

Theorem 2.4.29 ([Nys97, Theorem 6.2])**.** *Given* Ω *a* $\mathrm{Lip}(1, 1/2)$ *cylinder the following statements are equivalent:*

(1) $\omega \in A_\infty(\mathrm{d}\sigma)$.

(2) There exists a $1 < p < \infty$*, potentially large, such that the* (D_p) *condition holds. That is by theorem 2.4.9, for every* $f \in L^p(\partial\Omega, \mathrm{d}\sigma)$ *there exists a unique solution* u *to* (2.1.1) *such that* $u \to f$ *non-tangentially a.e. and* $\|Nu\|_{L^p(\partial\Omega, \mathrm{d}\sigma)} \lesssim \|f\|_{L^p(\partial\Omega, \mathrm{d}\sigma)}$.

(3) ω *is absolutely continuous with respect to* σ *and* $\omega \in B_{p'}(\mathrm{d}\sigma)$*, where* $1/p + 1/p' = 1$.

Remark 2.4.30. Theorem 2.4.23 states the class B_p is open therefore the property $(D)_p$ is open. Given a domain Ω and a parabolic equation L if $(D)_p$ holds then there exists ε (depending on Ω and L) such that $(D)_{p-\varepsilon}$ holds.

The following two results motivate and show the partial converse of theorem 4.1.2 later from [Riv03]. They were first shown for the heat equation in [Bro89a] and then extended to general parabolic PDE in [Nys97]. First the local inequality where $Lu = \mathrm{div}(A\nabla u) - u_t$.

Lemma 2.4.31. *Let* Ω *be a* $\mathrm{Lip}(1, 1/2)$ *cylinder and* $0 < p < \infty$*. Given* $\Delta_r \subset \partial\Omega$ *let* $Lu = 0$ *in* $T(\Delta_{2r})$ *with* $B = 0$*. If* $\omega \in A_\infty(\mathrm{d}\sigma)$ *then*

$$\int_{\Delta_r} S_a(u)^p \,\mathrm{d}\sigma \lesssim \int_{\Delta_r} N_b(u)^p \,\mathrm{d}\sigma \tag{2.4.18}$$

for $a < b$.

Proposition 2.4.32. *Let* $V_r(k)$ *be a corkscrew point of the ball* $\Delta_r(X,k) \subset \partial\Omega$*, where* Ω *is a* $\mathrm{Lip}(1, 1/2)$ *cylinder. Let* $0 < p < \infty$*, let* u *solve* $Lu = 0$ *in* Ω *with* $B = 0$ *and* $u(X,t) = 0$ *for all* $t < T_0$*, where* T_0 *is a fixed negative number. If* $\omega \in A_\infty(\mathrm{d}\sigma)$ *then*

$$\int_{\partial\Omega} N_a(u)^p \,\mathrm{d}\sigma \lesssim \int_{\partial\Omega} S_a(u)^p \,\mathrm{d}\sigma + \sum_{k=-\infty}^{\infty} r^{n+1} u(V_r(k)) \lesssim \int_{\partial\Omega} N_a(u)^p \,\mathrm{d}\sigma. \tag{2.4.19}$$

Perturbation Results

The elliptic perturbation theory from [FKP91] remains valid for parabolic equations in this setting by an adaptation of the elliptic proofs in [Swe98; Nys97]. The reason why we can obtain these perturbation results in $\mathrm{Lip}(1, 1/2)$ cylinders, when we know in general we can't solve the L^p Dirichlet problem in these domains [KW88; HL96], is because we have already assumed solvability.

Let

$$L_0 = \operatorname{div}(A_0(X,t)\nabla) - \partial_t \quad \text{and} \quad L_1 = \operatorname{div}(A_1(X,t)\nabla) - \partial_t$$

with associated parabolic measures ω_0 and ω_1 respectively; and let

$$a(X,t) = \sup_{B_{\delta(X,t)/2}(X,t)} |A_0(Y,s) - A_1(Y,s)|.$$

Theorem 2.4.33 ([Swe98; Nys97]). *Let $\Omega \subset \mathbb{R}^{n+1}$ be a $\mathrm{Lip}(1,1/2)$ cylinder and $\omega_0 \in B_p(\mathrm{d}\sigma)$. If*

$$\lim_{r\to 0^+} \sup_{(Y,s)\in\partial\Omega} \int_{\Delta_r(Y,s)} \frac{1}{\sigma(\Delta_r(Y,s))} \int_{\Gamma^r(Y,s)} \frac{a(X,t)^2}{\delta(X,t)^{n+2}} \,\mathrm{d}X\,\mathrm{d}t\,\mathrm{d}\sigma(Y,s) = 0 \tag{2.4.20}$$

then $\omega_1 \in B_p(\mathrm{d}\sigma)$.

If we do not assume the vanishing Carleson measure condition and just assume a Carleson measure condition then we may only conclude an A_∞ result.

Theorem 2.4.34 ([Nys97]). *Let $\Omega \subset \mathbb{R}^{n+1}$ be a $\mathrm{Lip}(1,1/2)$ cylinder and $\omega_0 \in A_\infty(\mathrm{d}\sigma)$. If*

$$\lim_{r\to 0^+} \sup_{(Y,s)\in\partial\Omega} \int_{\Delta_r(Y,s)} \frac{1}{\sigma(\Delta_r(Y,s))} \int_{\Gamma^r(Y,s)} \frac{a(X,t)^2}{\delta(X,t)^{n+2}} \,\mathrm{d}X\,\mathrm{d}t\,\mathrm{d}\sigma(Y,s) \leq C \tag{2.4.21}$$

then $\omega_1 \in A_\infty(\mathrm{d}\sigma)$.

2.5 Further Harmonic Analysis

2.5.1 Carleson Measures

Definition 2.5.1 (Carleson measure). *Let Ω be a $\mathrm{Lip}(1,1/2)$ cylinder. A measure μ on Ω is a Carleson measure if there exists a constant $C = C(d)$ such that for all surface balls Δ_r with $r \leq d$*

$$\mu(T(\Delta_r)) \leq C\sigma(\Delta_r). \tag{2.5.1}$$

The Carleson norm is the best possible constant C and is denoted by $\|\mu\|_{C,d}$. When the context is clear we drop the d and just write $\|\mu\|_C$. μ is a vanishing Carleson measure if $\|\mu\|_{C,d} \to 0$ as $d \to 0+$.

When we are working on U, the upper half space, we can reformulate the Carleson condition using parabolic boundary cubes $Q_r \subset \mathbb{R}^{n-1} \times \mathbb{R}$ and corresponding Carleson regions $T(Q_r)$. The Carleson condition (2.5.1) then becomes

$$\mu(T(Q_r)) \leq C|Q_r| = Cr^{n+1}, \tag{2.5.2}$$

with an analogous vanishing Carleson condition. The Carleson norms induced by (2.5.1) and (2.5.2) are not equal but are comparable.

Examples 2.5.2 ([Gra09]).

- In $\mathbb{R}^2_+$ in polar coordinates $\mathrm{d}r\,\mathrm{d}\theta$ is a Carleson measure but not a vanishing Carleson measure.

- In a bounded domain Ω the Lebesgue measure is a vanishing Carleson measure with norm $\|\mathrm{d}x\|_C = \operatorname{diam}(\Omega)$.

- Fix a line l in $\mathbb{R}^2$ and let $\mu(A) = \sigma(l \cap A)$ for σ the surface measure of l and any set A. Then μ is a Carleson measure but not a vanishing Carleson measure.

2.5.2 BMO and VMO

For a given $f : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}$ let

$$f_{Q_r} = \frac{1}{|Q_r|} \int_{Q_r} f(x,t) \,\mathrm{d}x \,\mathrm{d}t.$$

Definition 2.5.3 (BMO and VMO)**.** *A function f belongs to the parabolic version of the usual* BMO *(bounded mean oscillation) space with the norm $\|f\|_*$ if*

$$\|f\|_* = \sup_{Q_r} \frac{1}{|Q_r|} \int_{Q_r} |f - f_{Q_r}| \,\mathrm{d}x \,\mathrm{d}t < \infty. \tag{2.5.3}$$

We write $\|f\|_{*,d}$ to be the BMO *norm of f where the supremum in* (2.5.3) *is taken over all cubes Q_r with $r \le d$. Due to the localisation that happens in chapter 4, we choose to define the space* VMO *(vanishing mean oscillation) as the subspace of* BMO *functions such that $\|f\|_{*,d} \to 0$ as $d \to 0$.*

This definition implies that VMO is the closure of all continuous functions in the BMO norm [Sar75]. Alternatively, if we define

$$d(f, \mathrm{VMO}) := \inf_{h \in C} \|f - h\|_*$$

then $f \in \mathrm{VMO}$ if and only if $d(f, \mathrm{VMO}) = 0$; for $f \in \mathrm{BMO}$ this measures the distance of f to VMO.

Proposition 2.5.4 (Properties of BMO functions [Duo01; Gra09])**.** *Let $f, g \in \mathrm{BMO}$ then the following properties hold:*

(i) $\frac{1}{2}\|f\|_* \le \sup_Q \inf_{a \in \mathbb{R}} \frac{1}{|Q|} \int_Q |f - a| \,\mathrm{d}x \,\mathrm{d}t \le \|f\|_*$.

(ii) $L^\infty \subset \mathrm{BMO}$ *and* $\|f\|_* \le 2\|f\|_{L^\infty}$.

(iii) $\|f\|_* = 0$ *if and only if f is a.e. equal to a constant.*

(iv) The BMO *seminorm of f is invariant under translation and dilation.*

(v) $\|f + g\|_* \le \|f\|_* + \|g\|_*$ *and* $\|\lambda f\|_* = |\lambda| \|f\|_*$.

(vi) For all $1 \le p < \infty$

$$\|f\|_* \sim \sup_Q \left(\frac{1}{|Q|} \int_Q |f - f_Q|^p \,\mathrm{d}x \,\mathrm{d}t \right)^{1/p}.$$

From property (iii) above $\|\cdot\|_*$ is not a norm, it is only a seminorm. However, if we identify elements of BMO whose difference is just a constant then it becomes a norm.

Examples 2.5.5. By property (ii) in proposition 2.5.4 $L^\infty \subset \mathrm{BMO}$, however the opposite inclusion doesn't hold. The archetypal example on $\mathbb{R}^n$ is the function

$$f(x) = \begin{cases} \log\left(\frac{1}{|x|}\right) & |x| < 1, \\ 0 & |x| \ge 1. \end{cases}$$

In addition, this example shows that being in BMO is not just a consideration of size but also of cancellation — $\mathrm{sgn}(x) f(x) \notin \mathrm{BMO}$ even though its absolute value f is.

Similarly, we can define an unbounded function that belongs to $\mathrm{VMO}(\mathbb{R}^n)$

$$f(x) = \begin{cases} \log\log\left(\frac{1}{|x|}\right) & |x| < 1/e, \\ 0 & |x| \ge 1/e. \end{cases}$$

Both of these examples are the maximum rate of growth possible for BMO and VMO functions [Duo01].

Lemma 2.5.6 ([FS72; Str79])**.** *If $f \in \mathrm{BMO}$ then f satisfies the following growth condition for all $\varepsilon > 0$*

$$\int_{\mathbb{R}^n} \frac{|f(x,t)|}{1+\|(x,t)\|^{n+1+\varepsilon}}\, \mathrm{d}x\, \mathrm{d}t < \infty. \tag{2.5.4}$$

An important result is the duality between the Hardy space H^1 and BMO [FS72], which we use in the proof of theorem 4.2.7. H^1 is often used in singular integral theory as a replacement for the space L^1. Therefore, we very briefly introduce this space via its atomic decomposition and state the result. Further information and the plethora of definitions of H^1 can be found in [Ste93; Gra09] or other standard textbooks.

Definition 2.5.7 (Atom)**.** *An atom a is a function on $\mathbb{R}^n$ which is supported on a cube Q,*

$$\int_Q a = 0 \quad \textit{and} \quad \|a\|_{L^\infty} \le \frac{1}{|Q|}.$$

Definition 2.5.8 (H^1)**.** *The Hardy space H^1 is a linear combination of atoms*

$$H^1(\mathbb{R}^n) = \left\{ \sum_j \lambda_j a_j : a_j \textit{ atoms}, \lambda_j \in \mathbb{R}, \sum_j |\lambda_j| < \infty \right\}$$

with the norm

$$\|f\|_{H^1} = \inf \left\{ \sum_j |\lambda_j| : f = \sum_j \lambda_j a_j \right\}.$$

With this norm H^1 is a Banach space and $H^1 \subset L^1$.

We also need the following dense subclass of H^1 from [Ste70, p. 225].

Definition 2.5.9. *Let H^1_{00} consist of all functions $f \in H^1$ such that f is continuous and $f \in \mathcal{S}$. The space $\mathcal{S}$ is the Schwartz space (the class of rapidly decreasing smooth functions) — $f \in \mathcal{S}$ if fp is bounded for any polynomial p and every partial derivative of f is also continuous and rapidly decreasing.*

The following duality theorem was proved in [FS72; CT75; CT77; CW77] (the middle two references proved the duality on parabolic spaces and the last reference also examines spaces of homogeneous type).

Theorem 2.5.10 (H^1 and BMO duality)**.** *The dual of VMO is H^1 and the dual of H^1 is* BMO*.*

2.5.3 Relationships Between Carleson Measures, Non-tangential Maximal Functions, BMO and A_∞ Weights

All of these following relationships were proved in the isotropic (elliptic) setting however the proofs in the parabolic case are identical. Recall that $U = \{(x_0, x, t) : x_0 > 0\}$ is the upper half space in $\mathbb{R}^{n+1}$.

Theorem 2.5.11 (Duality statement between Carleson measures and non-tangential maximal functions [Ste93, p. 59])**.** *Let μ be a Carleson measure, U the upper half space and $0 < p < \infty$ then for any function $u : U \to \mathbb{R}$ we have*

$$\int_U |u|^p \, \mathrm{d}\mu \le \|\mu\|_C \|N(u)\|^p_{L^p}, \tag{2.5.5}$$

with a local version holding on Carleson boxes as well.

Let $\psi_{x_0}(x,t) = x_0^{-(n+1)} \psi(x/x_0, t/x_0^2)$.

Theorem 2.5.12 ([Duo01, Theorem 9.5; Gra09, Theorem 7.3.7])**.** *Suppose that ψ is a bounded, integrable, positive, radial and decreasing function. Then a measure μ is Carleson if and only if for every $1 < p < \infty$*

$$\int_U |\psi_{x_0} * f(x,t)|^p \, \mathrm{d}\mu(x_0,x,t) \leq C \int_{\mathbb{R}^n} |f(x,t)|^p \, \mathrm{d}x \, \mathrm{d}t. \tag{2.5.6}$$

The constant C is comparable with $\|\mu\|_C$.

This characterises Carleson measures as measures for which the Poisson integral in the x_0 direction defines a bounded operator from $L^p(\mathbb{R}^n, \mathrm{d}x\,\mathrm{d}t)$ to $L^p(U, \mathrm{d}\mu)$.

Definition 2.5.13 ([CT75; Ste93])**.** *We say a function $\psi \in \mathcal{S}$ is non-degenerate if $\hat{\psi}$ does not vanish identically on any (parabolic) ray from the origin. That is $\hat{\psi}(r\xi, r^2\tau)$ does not vanish identically in r for $(\xi,\tau) \neq (0,0)$.*

Theorem 2.5.14 ([Ste93, Chapter IV, §4], c.f. [Str80])**.** *Let $\psi \in \mathcal{S}(\mathbb{R}^{n-1} \times \mathbb{R})$ such that $\int \psi = 0$ and $f \in \mathrm{BMO}$. Then the measure μ defined by*

$$\mathrm{d}\mu = |\psi_{x_0} * f|^2 \frac{\mathrm{d}x_0 \, \mathrm{d}x \, \mathrm{d}t}{x_0} \tag{2.5.7}$$

is a Carleson measure on U with $\|\mu\|_C \lesssim \|f\|_*^2$.

Conversely, suppose ψ is non-degenerate and the growth condition (2.5.4) *holds then if* (2.5.7) *is a Carleson measure with norm $\|\mu\|_C$ then $f \in \mathrm{BMO}$ and $\|f\|_*^2 \lesssim \|\mu\|_C$.*

Remark 2.5.15. If we assume ψ has compact support then the same result holds without assuming the growth condition [Str80]. Furthermore when ψ has compact support then the same result holds for vanishing Carleson measures and VMO [LPW15].

An interesting open problem would be to obtain the vanishing Carleson measures and VMO result when ψ does not have compact support but instead satisfies the following conditions (along with non-degeneracy)

$$\begin{aligned} |\psi_{x_0}(x,t)| &\lesssim \frac{x_0}{(x_0 + \|(x,t)\|^{n+2})}, \\ |\psi_{x_0}(x,t) - \psi_{x_0}(y,s)| &\lesssim \frac{\|(x,t)-(y,s)\|^\delta}{(x_0 + \|(x,t)\|^{n+2+\delta})} \quad \text{if } 2\|(x,t)-(y,s)\| < \|(x,t)\|, \end{aligned} \tag{2.5.8}$$

where $0 < \delta \leq 1$. To the best of our knowledge this is unknown even in the easier direction and in the isotropic setting. This would probably involve a VMO-type study similar to the BMO one given in [Str80, §2] and/or via wavelets.

This convolution appears in (4.2.60) and (4.2.62) in the proof of lemma 4.2.26, see [Hof97, Lemma 1; HL96, Lemma 2.8].

Remark 2.5.16. We in essence use a more nuanced version of theorem 2.5.14 in the proof of theorem 4.2.7, where for the converse direction instead of using one convolution ψ we use a finite family ψ^k of them. As a family they satisfy a non-degeneracy condition but we allow some of our family ψ^k to be degenerate as long as they aren't all degenerate along the same ray. See the proof of theorem 4.2.7 and [Str80, Theorem 2.5] for details.

By an observation of the proof of theorem 2.5.14 we have the following lemma.

Lemma 2.5.17 ([FS72; Ste93, Chapter IV, §4.3.3])**.** *Let ψ be as in theorem 2.5.14 above and $f \in \mathrm{BMO}$ then*

$$\sup_{x_0>0} \|\psi_{x_0} * f\|_{L^\infty} \lesssim \|f\|_*, \tag{2.5.9}$$

with the implicit constant depending on ψ.

From the reverse Jensen's inequality, condition (6) of theorem 2.4.25, and the John-Nirenberg inequality for BMO functions [Duo01] there is a relationship between weights and BMO functions.

Theorem 2.5.18 ([Ste93, Chapter V, §6.2])**.** *A function $f \in \mathrm{BMO}$ if and only if $f = c\log w$ for some weight $w \in A_\infty$.*

2.5.4 Calderòn-Zygmund Operators

We start by defining Calderòn-Zygmund operators on $\mathbb{R}^n$ with the usual Euclidean homogeneity and norm.

Definition 2.5.19 (Standard kernel [MC97])**.** *Let $\triangle$ denote the diagonal of $\mathbb{R}^n \times \mathbb{R}^n$, that is $\triangle = \{(x,x) : x \in \mathbb{R}^n\}$. We say that $K : \mathbb{R}^n \times \mathbb{R}^n \setminus \triangle \to \mathbb{R}$ is a standard Calderòn-Zygmund kernel if there exists $\delta > 0$ such that*

$$\begin{aligned} |K(x,y)| &\lesssim \frac{1}{|x-y|^n}, \\ |K(x,y) - K(x,z)| &\lesssim \frac{|y-z|^\delta}{|x-y|^{n+\delta}} \quad \text{if} \quad |x-y| > 2|y-z|, \\ |K(x,y) - K(x,z)| &\lesssim \frac{|x-w|^\delta}{|x-y|^{n+\delta}} \quad \text{if} \quad |x-y| > 2|x-w|. \end{aligned} \tag{2.5.10}$$

Definition 2.5.20. *An operator T is a Calderòn-Zygmund operator with a standard kernel if:*

(i) T is bounded on $L^2(\mathbb{R}^n)$ and

(ii) there exists a standard kernel K such that if $f \in L^2$ with compact support then

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y)\,\mathrm{d}y, \quad x \notin \operatorname{supp}(f). \tag{2.5.11}$$

Theorem 2.5.21 ([MC97])**.** *If T is a Calderòn-Zygmund operator then $T : L^p \to L^p$ for all $1 < p < \infty$, $T : H^1 \to L^1$ and $T : \mathrm{BMO} \to L^\infty$.*

Definition 2.5.22. *We say a Calderòn-Zygmund operator T satisfies the condition $T(1) = 0$ if $\int_{\mathbb{R}^n} T^*(a)\,\mathrm{d}x = 0$ for each atom $a \in H^1$ and where T^* is the adjoint operator. Similarly $T^*(1) = 0$ if $\int_{\mathbb{R}^n} T(a)\,\mathrm{d}x = 0$ for each atom $a \in H^1$.*

Theorem 2.5.23 ([MC97, Chapter 7, Theorem 3])**.** *A Calderòn-Zygmund operator $T : L^2 \to L^2$ defines a continuous linear operator on H^1 if and only if $T^*(1) = 0$. Moreover by duality, it defines a continuous linear operator on* BMO *if and only if $T(1) = 0$.*

The space that we're working on $(\mathbb{R}^{n-1} \times \mathbb{R}, \|\cdot\|)$ is a very simple case of a space of homogeneous type introduced by [CW71]. Therefore we can call upon that theory (using a very big hammer to crack a small nut) to show that theorems 2.5.21 and 2.5.23 above hold in our setting. For the operators that we apply these theorems to, we could have just used the results from [FR66; FR67] for $L^p \to L^p$ bounds for Calderòn-Zygmund multiplier operators with inhomogeneous dilations, and the BMO $\to$ BMO result from [Pee66][1]. However, we give the theory for generalised Calderòn-Zygmund operators on $(\mathbb{R}^{n-1} \times \mathbb{R}, \|\cdot\|)$ for a more complete presentation in the hope that the reader may find the more general results useful. For the full theory on spaces of homogeneous type see the works cited below.

For the following section let $z, u, v, w \in \mathbb{R}^{n-1} \times \mathbb{R}$, i.e. $z = (x,t)$.

Definition 2.5.24 (Standard parabolic kernel [DH09])**.** *Again let $\triangle$ denote the diagonal, that is $\triangle = \{(z,z) : z \in \mathbb{R}^{n-1} \times \mathbb{R}\}$. We say that $K : \mathbb{R}^n \times \mathbb{R}^n \setminus \triangle \to \mathbb{R}$ is a standard parabolic Calderòn-Zygmund kernel if there exists $\delta > 0$ such that*

$$\begin{aligned} |K(z,u)| &\lesssim \frac{1}{\|z-u\|^{n+1}}, \\ |K(z,u) - K(z,v)| &\lesssim \frac{\|u-v\|^\delta}{\|z-u\|^{n+1+\delta}} \quad \text{if} \quad \|z-u\| > 2\|u-v\|, \\ |K(z,u) - K(w,u)| &\lesssim \frac{\|z-w\|^\delta}{\|z-u\|^{n+1+\delta}} \quad \text{if} \quad \|z-u\| > 2\|z-w\|. \end{aligned} \tag{2.5.12}$$

[1] We would really need an extension of this result to the parabolic setting for the multiplier operators we apply it to. However, this is easily done, c.f. [Pee66, Remark 1.2 and 1.3].

We also have the analogous version of definition 2.5.20 for parabolic Calderòn-Zygmund operators.

Theorem 2.5.25 ([CW71; DH09])**.** *If T is a parabolic Calderòn-Zygmund operator then $T : L^p \to L^p$ for all $1 < p < \infty$, $T : H^1 \to L^1$ and $T : \mathrm{BMO} \to L^\infty$.*

Theorem 2.5.26 ([DH09, Theorem 4.27])**.** *A parabolic Calderòn-Zygmund operator $T : L^2 \to L^2$ defines a continuous linear operator on H^1 if and only if $T^*(1) = 0$. Moreover by duality, it defines a continuous linear operator on* BMO *if and only if $T(1) = 0$.*

We now give a version of these theorems in the form that we use them.

Corollary 2.5.27. *Let $Tf = K * f$ be a multiplier operator with kernel $K(x,t)$ for $f : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}$. Let T be homogeneous of degree $-(n+1)$ with respect to the parabolic scaling (or $\widehat{K}$ homogeneous of degree 0), $K(\lambda x, \lambda t) = \lambda^{-(n+1)} K(x,t)$, and let K have zero average on spheres around the origin (with respect to the parabolic weight for polar coordinates, see* (2.1.8)*). The standard parabolic kernel estimates for K with $\delta = 1$ become (for $z, v \in \mathbb{R}^{n-1} \times \mathbb{R}$)*

$$\begin{aligned} |K(z)| &\lesssim \frac{1}{\|z\|^{n+1}}, \\ |K(z) - K(v)| &\lesssim \frac{\|z - v\|}{\|z\|^{n+2}} \quad \text{if} \quad \|z\| > 2\|z - v\|. \end{aligned} \tag{2.5.13}$$

If K satisfies (2.5.13) *then by theorems 2.5.25 and 2.5.26 T is bounded on L^p for all $1 < p < \infty$, $T : H^1 \to H^1$, preserves the class H^1_{00}, and $T : \mathrm{BMO} \to \mathrm{BMO}$.*

Note that since K has zero average on spheres we may implicitly use the $T1$ theorem to deduce $L^2 \to L^2$ bounds even if $|\widehat{K}|$ is unbounded. See also [FS72, Chapter 2, §3] for similar results to this corollary.

Example 2.5.28 (Parabolic Riesz transforms)**.** Let R_j be the parabolic Riesz transforms then we would hope that, as usual, they are the archetypal Calderòn-Zygmund operators. Let

$$\begin{aligned} \widehat{R_j}(\xi, \tau) &= \frac{i\xi_j}{\|(\xi, \tau)\|} \quad \text{for } 1 \le j \le n-1, \\ \widehat{R_n}(\xi, \tau) &= \frac{\tau}{\|(\xi, \tau)\|^2}. \end{aligned} \tag{2.5.14}$$

Then each R_j satisfies the assumptions of corollary 2.5.27 and hence each R_j is a bounded operator on L^p for all $1 < p < \infty$, $R_j : H^1 \to H^1$, preserves the class H^1_{00} and $R_j : \mathrm{BMO} \to \mathrm{BMO}$.

Chapter 3

The Regularity Problem

This chapter is based upon work in [DD17] and is organised as follows. In section 3.2 we introduce the $L^p_{1,1/2}$ parabolic Sobolev space on $\mathbb{R}^n$ and domains, and prove the consistency of the definition. In section 3.3 we state a Poincaré type inequality and prove theorem 3.1.1.

3.1 Introduction

We study the relationship between the solvability of the Regularity and the Dirichlet boundary value problems for parabolic equations for

$$\begin{cases} u_t = \operatorname{div}(A\nabla u) & \text{in } \Omega \subset \mathbb{R}^{n+1}, \\ u = f & \text{on } \partial\Omega, \end{cases} \tag{3.1.1}$$

on $\operatorname{Lip}(1, 1/2)$ cylinders Ω introduced in definition 2.1.3. These domains are bounded and Lipschitz in the spatial variables; and unbounded and $\operatorname{Lip}_{1/2}$ in time. Furthermore, we assume that the matrix $A(X, t)$ satisfies the ellipticity condition, and its coefficients are bounded and measurable, see (1.0.2).

Our result in this chapter proves that if the Regularity problem (R_p) for the operator L on the domain Ω is solvable for some $1 < p < \infty$ then the Dirichlet problem $(D^*)_{p'}$ for the adjoint

$$\begin{cases} -u_t = \operatorname{div}(A^*\nabla u) & \text{in } \Omega \subset \mathbb{R}^{n+1}, \\ \quad u = f & \text{on } \partial\Omega, \end{cases} \tag{3.1.2}$$

is also solvable on the domain Ω. See definitions 2.4.6 and 2.4.7 for the definition of $(D)_p$ and $(R)_p$.

Note $L^* = \operatorname{div}(A^*\nabla) + \partial_t$ is a backward in time parabolic operator. As in the introduction (chapter 1), by the reflection in time we see that $L^*u = 0$ on Ω is equivalent to

$$\tilde{L}v = \operatorname{div}(\tilde{A}^*\nabla v) - v_t = 0 \qquad \text{on } \tilde{\Omega}, \tag{3.1.3}$$

where $\tilde{\Omega}$ is the reflection of Ω in the t variable i.e. $\tilde{\Omega} = \{(X, -t) : (X, t) \in \Omega\}$. Hence, the solvability of the $L^{p'}$ Dirichlet problem for the operator L^* on Ω is equivalent to the solvability of the $L^{p'}$ Dirichlet problem for the operator $\tilde{L}$ on $\tilde{\Omega}$. Here $\tilde{L}v = 0$ is the usual forward in time parabolic PDE.

Our result is motivated by the analogous result in the elliptic setting by [KP93] where, amongst other relationships, they show that (R_p) implied $(D^*)_{p'}$ for elliptic operators $\operatorname{div}(A\nabla\cdot)$ in bounded Lipschitz domains. This has been observed for some specific parabolic PDE (such as the heat equation and constant coefficient systems, [HL96, p. 418; Nys06] respectively) in more restrictive Lewis-Murray type time-varying domains. Recall from the introduction in chapter 1, these are domains which are $\operatorname{Lip}(1, 1/2)$ and half a derivative in time lives in parabolic BMO — see section 4.2 for details. Nyström [Nys06] also shows that no duality can be expected between Dirichlet and Neumann boundary value problems in non-smooth time-varying domains.

In our result we remove any restrictions on the coefficients of the scalar elliptic operator (beyond the ellipticity hypothesis) and establish the result on the largest reasonable class of domains. It is worth pointing out that due to the roughness of the coefficients and of the boundary of these domains the usual techniques (such as layer potentials and Fourier methods) are not available.

Theorem 3.1.1. *Let Ω be a* $\mathrm{Lip}(1,1/2)$ *cylinder, as in definition 2.1.3, with character (ℓ, N, d); and let $A(X,t)$ be bounded, measurable and elliptic. If the Regularity problem $(R)_p$ is solvable for the equation*

$$\begin{cases} u_t = \operatorname{div}(A\nabla u) & \text{in } \Omega \subset \mathbb{R}^{n+1}, \\ u = f & \text{on } \partial\Omega, \end{cases} \tag{3.1.4}$$

for some $1 < p < \infty$ then the Dirichlet problem $(D^*)_{p'}$ is solvable for the adjoint equation

$$\begin{cases} -u_t = \operatorname{div}(A^*\nabla u) & \text{in } \Omega, \\ \quad u = f & \text{on } \partial\Omega, \end{cases} \tag{3.1.5}$$

where $p' = p/(p-1)$.

3.2 Parabolic Sobolev Spaces

When considering the appropriate function space for our boundary data we want it to have the same homogeneity as the PDE. As a rule of thumb, one derivative in time behaves like two derivatives in space. If we impose data with one derivative in the spatial variables then the correct order of our time derivative should be 1/2. This problem has been studied previously in [HL96; HL99; Nys06], who have followed [FJ68] in defining the homogeneous parabolic Sobolev space $\dot{L}^p_{1,1/2}$ as below. In section 4.2.2 we study the end point characteristics of the following operators and give some equivalent definitions.

Definition 3.2.1. *The homogeneous parabolic Sobolev space $\dot{L}^p_{1,1/2}(\mathbb{R}^n)$, for $1 < p < \infty$, is defined to consist of an equivalence class of functions f with distributional derivatives satisfying $\|f\|_{\dot{L}^p_{1,1/2}(\mathbb{R}^n)} < \infty$, where*

$$\|f\|_{\dot{L}^p_{1,1/2}(\mathbb{R}^n)} = \|\mathbb{D}f\|_{L^p(\mathbb{R}^n)} \tag{3.2.1}$$

and $\mathbb{D}$ is the parabolic derivative (in space and time)

$$\widehat{\mathbb{D}f}(\xi,\tau) := \|(\xi,\tau)\|\widehat{f}(\xi,\tau), \tag{3.2.2}$$

where ξ and τ denote the spatial and temporal variables on the Fourier side respectively. Recall $\|(x,t)\| = |x| + |t|^{1/2}$ denotes the parabolic norm.

We also define the inhomogeneous parabolic Sobolev space $L^p_{1,1/2}(\mathbb{R}^n)$ as an equivalence class of functions f with distributional derivatives satisfying $\|f\|_{L^p_{1,1/2}(\mathbb{R}^n)} < \infty$, where

$$\|f\|_{L^p_{1,1/2}(\mathbb{R}^n)} = \|\mathbb{D}f\|_{L^p(\mathbb{R}^n)} + \|f\|_{L^p(\mathbb{R}^n)}. \tag{3.2.3}$$

Other authors [Bro89b; Bro90; HL99; Mit01; Nys06; CRS15] have only considered either Lipschitz cylinders or graph domains and so have only needed to control the homogeneous norm. However, because we are considering an infinite time-varying cylinder made from a local collection of graphs ϕ_j we need to have additional control over the L^p norm of f to control terms that arise from taking a smooth partition of unity.

In addition, following [FR67], we define the *parabolic half derivative in time* by

$$\widehat{\mathbb{D}_n f}(\xi,\tau) := \frac{\tau}{\|(\xi,\tau)\|}\widehat{f}(\xi,\tau). \tag{3.2.4}$$

By parabolic singular integral theory [FR66; FR67], see corollary 2.5.27, we have

$$\|\mathbb{D}f\|_{L^p(\mathbb{R}^n)} \sim \|\mathbb{D}_n f\|_{L^p(\mathbb{R}^n)} + \|\nabla f\|_{L^p(\mathbb{R}^n)}. \tag{3.2.5}$$

One small result of this thesis is that we have another characterisation of the spaces $\dot{L}^p_{1,1/2}(\mathbb{R}^n)$ and $L^p_{1,1/2}(\mathbb{R}^n)$ by an equivalent norm. To motivate this norm, if we apply Plancherel's theorem for $p = 2$ we have

$$\|\mathbb{D}f\|_{L^2(\mathbb{R}^n)} \sim \|D^t_{1/2} f\|_{L^2(\mathbb{R}^n)} + \|\nabla f\|_{L^2(\mathbb{R}^n)}, \tag{3.2.6}$$

where $D^t_{1/2}$ denotes the one-dimensional half derivative of f in the time variable. We show in theorem 3.2.3 that this equivalence holds for all $1 < p < \infty$.

If $0 < \alpha \leq 2$, then for $g \in C_0^\infty(\mathbb{R})$ the *one-dimensional fractional differentiation operators* D_α are defined by

$$\widehat{D_\alpha g}(\tau) := |\tau|^\alpha \widehat{g}(\tau). \tag{3.2.7}$$

It is also well known [Ste93] if $0 < \alpha < 1$ then

$$D_\alpha g(s) = c \int_{\mathbb{R}} \frac{g(s) - g(\tau)}{|s - \tau|^{1+\alpha}} \,\mathrm{d}\tau \tag{3.2.8}$$

whenever $s \in \mathbb{R}$. If $h(x,t) \in C_0^\infty(\mathbb{R}^n)$ then by $D^t_\alpha h : \mathbb{R}^n \to \mathbb{R}$ we mean the function $D_\alpha h(x, \cdot)$ defined a.e. for each fixed $x \in \mathbb{R}^{n-1}$. We call $D^t_{1/2} h$ the *pointwise half derivative in time*. Since $\frac{|\tau|^{1/2}}{\|(\xi,\tau)\|}$ is an L^p multiplier for $1 < p < \infty$ [Ste70, Theorem 6, p. 109] (or corollary 2.5.27) we have the following bound.

Lemma 3.2.2. *Let $f : \mathbb{R}^n \to \mathbb{R}$ and $1 < p < \infty$ then*

$$\|D^t_{1/2} f\|_{L^p(\mathbb{R}^n)} \lesssim \|\mathbb{D}f\|_{L^p(\mathbb{R}^n)}. \tag{3.2.9}$$

Theorem 3.2.3. *Let $f : \mathbb{R}^n \to \mathbb{R}$ and $1 < p < \infty$ then*

$$\|\mathbb{D}_n f\|_{L^p(\mathbb{R}^n)} \lesssim \|D^t_{1/2} f\|_{L^p(\mathbb{R}^n)} + \|\nabla f\|_{L^p(\mathbb{R}^n)}. \tag{3.2.10}$$

Therefore $\|f\|_{\dot{L}^p_{1,1/2}(\mathbb{R}^n)} = \|\mathbb{D}f\|_{L^p(\mathbb{R}^n)} \sim \|D^t_{1/2} f\|_{L^p(\mathbb{R}^n)} + \|\nabla f\|_{L^p(\mathbb{R}^n)}$ *for* $1 < p < \infty$ *and so*

$$\|f\|_{L^p_{1,1/2}(\mathbb{R}^n)} \sim \|D^t_{1/2} f\|_{L^p(\mathbb{R}^n)} + \|\nabla f\|_{L^p(\mathbb{R}^n)} + \|f\|_{L^p(\mathbb{R}^n)}.$$

The proof uses the same approach as [HL96, Section 7] to obtain L^p bounds instead of their mixed BMO and L^∞ bounds.

Proof. By approximation we may assume that $f \in C_0^\infty(\mathbb{R}^n)$. Also we can assume $f(0) = 0$ by replacing f by $f - f(0)$ and noting that $\mathbb{D}_n$ and $D^t_{1/2}$ map constants to the 0 element. Let

$$m(\xi, \tau) = \frac{\tau}{|\tau|^{1/2} \|(\xi, \tau)\|}$$

then we have

$$\widehat{\mathbb{D}_n f}(\xi, \tau) = \hat{f}(\xi, \tau) \left(m(\xi, \tau) |\tau|^{1/2} \right)$$

for $(\xi, \tau) \in \mathbb{R}^n$.

This multiplier m is not smooth enough to apply standard multiplier theorems to. So as in [HL96], we use a smooth cut off function η to split this multiplier in two. Let $\eta \in C_0^\infty(\mathbb{R})$ be an even function with $\eta = 1$ on $(-3/2, -1/2)$ and $(1/2, 3/2)$; supported in $(-2, -1/4)$ and $(1/4, 2)$; and choose η such that $|D^k \eta| \lesssim 2^k$ for $0 \leq k \leq n + 4$. Let

$$m^+(\xi, \tau) = m(\xi, \tau) \eta \left(\frac{\tau}{\|(\xi, \tau)\|^2} \right)$$

and
$$m^{++}(\xi,\tau) = \frac{|\tau|^{1/2} m(\xi,\tau)\|(\xi,\tau)\|}{|\xi|^2}(1-\eta)\left(\frac{\tau}{\|(\xi,\tau)\|^2}\right)$$
then
$$\widehat{\mathbb{D}_n f}(\xi,\tau) = \hat{f}(\xi,\tau)\left(m^+(\xi,\tau)|\tau|^{1/2} + \frac{|\xi|^2}{\|(\xi,\tau)\|} m^{++}(\xi,\tau)\right).$$

Let $m_j^{++}(\xi,\tau) = \frac{\xi_j}{\|(\xi,\tau)\|} m^{++}(\xi,\tau)$ for $0 \le j \le n-1$ then we show there exist singular integral operators $T_{m_j^{++}}$ and T_{m^+} corresponding to m_j^{++} and m^+ respectively such that

$$\mathbb{D}_n f = cT_{m^+}(D^t_{1/2} f) + c\sum_{j=0}^{n-1} T_{m_j^{++}}(\partial_j f). \tag{3.2.11}$$

All we have to show is that T_{m^+} and $T_{m_j^{++}}$ exist and map L^p into L^p for $1 < p < \infty$.

First we consider m^+, which is infinitely differentiable away from the origin. It is not hard to show that if γ is a multi-index and a a non-negative integer then

$$|\partial_\xi^\gamma \partial_\tau^a m^+(\xi,\tau)| \lesssim \|(\xi,\tau)\|^{-(|\gamma|+2a)}, \tag{3.2.12}$$

for $1 \le a + |\gamma| \le n+4$, and that $|m^+(\xi,\tau)| \lesssim 1$. By singular integral with mixed homogeneity theory [FR66, p. 28] and corollary 2.5.27 we have that T_{m^+} exists and is bounded on L^p for $1 < p < \infty$.

Similarly considering m_j^{++}, by [HL96, (7.10)-(7.11)] we have

$$|\partial_\xi^\gamma \partial_\tau^a m_j^{++}(\xi,\tau)| \lesssim |\tau|^{1/2-a}\|(\xi,\tau)\|^{-(|\gamma|+1)}, \tag{3.2.13}$$

for $0 \le a + |\gamma| \le n+4$ and that the support of m_j^{++} is contained in

$$\left\{(\xi,\tau) : 0 \le |\tau| \le \|(\xi,\tau)\|^2/2\right\}. \tag{3.2.14}$$

Using these $|m_j^{++}(\xi,\tau)| \lesssim 1$ and by the same argument as before $T_{m_j^{++}}$ exists and is bounded on L^p for $1 < p < \infty$. □

So far we have only studied this parabolic Sobolev space $L^p_{1,1/2}$ on $\mathbb{R}^n$ however our aim is to work on the boundary $\partial\Omega$ where Ω is a $\mathrm{Lip}(1,1/2)$ cylinder.

Definition 3.2.4 (Parabolic Sobolev spaces on $\mathrm{Lip}(1,1/2)$ cylinders)**.** *Let $1 < p < \infty$ and Ω be a $\mathrm{Lip}(1,1/2)$ cylinder, as in definition 2.1.3, with pullback mappings $\rho_j : Q_{8d} \to \partial\Omega \cap 8\mathbb{Z}_j$, for $Q_{8d} \subset \mathbb{R}^{n-1}\times\mathbb{R}$. Let η_j be a smooth partition of unity of $\partial\Omega$ with the following properties:*

(i) $0 \le \eta_j \le 1$.

(ii) $\sum \eta_j = 1$.

(iii) The η_j's have bounded overlap: i.e. for each fixed (x,t) $\#\{j : \eta_j(x,t) > 0\} \le M$.

(iv) $\operatorname{supp} \eta_j \subset B_j(x_j,t_j)$ *with $r_j \sim d$, where d is from definition 2.1.3.*

We then define the $L^p_{1,1/2}$ norm on $\partial\Omega$ as

$$\|f\|_{L^p_{1,1/2}(\partial\Omega)} = \left(\sum_j \left(\|\mathbb{D}\left((f\eta_j)\circ\rho_j\right)\|^p_{L^p(\mathbb{R}^n)} + \|(f\eta_j)\circ\rho_j\|^p_{L^p(\mathbb{R}^n)}\right)\right)^{1/p}. \tag{3.2.15}$$

By the relationship in theorem 3.2.3 this is equivalent to

$$\|f\|^p_{L^p_{1,1/2}(\partial\Omega)} \sim \sum_j \left(\|\nabla\left((f\eta_j)\circ\rho_j\right)\|^p_{L^p(\mathbb{R}^n)} + \|D^t_{1/2}\left((f\eta_j)\circ\rho_j\right)\|^p_{L^p(\mathbb{R}^n)} + \|(f\eta_j)\circ\rho_j\|^p_{L^p(\mathbb{R}^n)}\right). \tag{3.2.16}$$

This definition in intuitive but it would be a poor choice if it isn't consistent when $\partial\Omega = \mathbb{R}^n$. We show in the following proposition that the parabolic Sobolev spaces defined by these two different ways have equivalent norms.

Proposition 3.2.5 (Consistency of the definitions of $L^p_{1,1/2}$ spaces)**.** *Let $\partial\Omega = \mathbb{R}^n$ then for all $f \in L^p_{1,1/2}(\partial\Omega)$ the $L^p_{1,1/2}$ spaces and norms defined in definition 3.2.1 and definition 3.2.4 are comparable.*

Proof. We want to show

$$\|f\|^p_{L^p_{1,1/2}(\mathbb{R}^n)} \sim \sum_j \left(\|\nabla(f\eta_j)\|^p_{L^p(\mathbb{R}^n)} + \|D^t_{1/2}(f\eta_j)\|^p_{L^p(\mathbb{R}^n)} + \|f\eta_j\|^p_{L^p(\mathbb{R}^n)} \right). \tag{3.2.17}$$

By theorem 3.2.3 we reduce this down to showing the following three properties:

(i)

$$\|f\|^p_{L^p(\mathbb{R}^n)} \sim \sum_j \|f\eta_j\|^p_{L^p(\mathbb{R}^n)}. \tag{3.2.18}$$

(ii)

$$\|\nabla f\|^p_{L^p(\mathbb{R}^n)} \leq \sum_j \|\nabla(f\eta_j)\|^p_{L^p(\mathbb{R}^n)} \lesssim \|\nabla f\|^p_{L^p(\mathbb{R}^n)} + \|f\|^p_{L^p(\mathbb{R}^n)}. \tag{3.2.19}$$

(iii)

$$\|D^t_{1/2} f\|^p_{L^p(\mathbb{R}^n)} + \|f\|^p_{L^p(\mathbb{R}^n)} \sim \sum_j \|D^t_{1/2}(f\eta_j)\|^p_{L^p(\mathbb{R}^n)} + \|f\|^p_{L^p(\mathbb{R}^n)}. \tag{3.2.20}$$

Let η_j be a partition of unity of $\mathbb{R}^n$ with each η_j supported in a unit ball $B_j = B_1(r_j, t_j)$.

Step 1: Proof of (3.2.18).

$f^p = \left(\sum_j f\eta_j\right)^p$ so by the triangle inequality $|f|^p \leq \sum_j |f\eta_j|^p$. Integrating both sides and using the monotone convergence theorem gives

$$\int |f|^p \leq \int \sum_j |f\eta_j|^p = \sum_j \int |f\eta_j|^p.$$

For the other direction, $|f\eta_j|^p \leq |f|^p$ so by the bounded overlap property of η_j we have $\sum_j |f\eta_j|^p \leq M|f|^p$. Integrating and using the monotone convergence theorem again gives

$$\sum_j \int |f\eta_j|^p = \int \sum_j |f\eta_j|^p \leq M \int |f|^p.$$

Step 2: Proof of (3.2.19).

Clearly $\sum_j \nabla(f\eta_j) = \sum_j \eta_j \nabla f + f \sum_j \nabla\eta_j$ and since $\sum_j \eta_j = 1$ then $\sum_j \nabla\eta_j = 0$. Therefore $\sum_j \nabla(f\eta_j) = \sum_j \eta_j \nabla f = \nabla f$. As in step 1, we have

$$\int |\nabla f|^p = \int \left| \sum_j \nabla(f\eta_j) \right|^p \leq \int \sum_j |\nabla(f\eta_j)|^p = \sum_j \int |\nabla(f\eta_j)|^p.$$

For the other direction we have $|\nabla(f\eta_j)|^p \leq |\nabla f|^p + |\nabla\eta_j|^p |f|^p$. Now $|\nabla\eta_j| \sim 1/d \sim 1$, therefore by the monotone convergence theorem

$$\sum_j \int |\nabla(f\eta_j)|^p = \int \sum_j |\nabla(f\eta_j)|^p \leq M \int |\nabla f|^p + M \int |f|^p.$$

So

$$\|\nabla f\|^p_{L^p(\mathbb{R}^n)} \le \sum_j \|\nabla(f\eta_j)\|^p_{L^p(\mathbb{R}^n)} \lesssim \|\nabla f\|^p_{L^p(\mathbb{R}^n)} + \|f\|^p_{L^p(\mathbb{R}^n)}.$$

Also by (3.2.18)

$$\sum_j \left(\|\nabla(f\eta_j)\|^p_{L^p(\mathbb{R}^n)} + \|f\eta_j\|^p_{L^p(\mathbb{R}^n)}\right) \sim \|\nabla f\|^p_{L^p(\mathbb{R}^n)} + \|f\|^p_{L^p(\mathbb{R}^n)}.$$

Step 3: Proof of (3.2.20).

We want to show the following two statements:

$$\sum_j \int |D^t_{1/2}(f\eta_j)|^p \lesssim \int |D^t_{1/2} f|^p + \int |f|^p, \tag{3.2.21}$$

$$\int |D^t_{1/2} f|^p \lesssim \sum_j \int |D^t_{1/2}(f\eta_j)|^p + \sum_j \int |f\eta_j|^p. \tag{3.2.22}$$

First let's tackle (3.2.21)

$$\sum_j \int_{\mathbb{R}^n} |D^t_{1/2}(f\eta_j)|^p \lesssim \int |D^t_{1/2} f|^p + \int |f|^p.$$

Recall that

$$D^t_{1/2}(f\eta_j)(x,t) = \int_{\mathbb{R}} \frac{f(x,s)\eta_j(x,s) - f(x,t)\eta_j(x,t)}{|s-t|^{3/2}}\,\mathrm{d}s.$$

Step 3.a: $D^t_{1/2}(f\eta_j)(x,t)$ for $(x,t) \notin 2B_j$.

We start by considering $D^t_{1/2}(f\eta_j)(x,t)$ when (x,t) is far away from the support of η_j. Let $A_k = \{(x,t) : \operatorname{dist}((x,t), B_j) \sim 2^k\}$, where dist is the parabolic distance. If $(x,t) \in A_k$ and $(x,s) \in \operatorname{supp} \eta_j$ then $\eta_j(x,t) = 0$ and $|t-s| > 2^{2k}$. Therefore

$$\begin{aligned}
\|D^t_{1/2}(f\eta_j)\|_{L^p(A_k)} &= \left(\int_{A_k} \left|\int_{\mathbb{R}} \frac{f(x,s)\eta_j(x,s)}{|s-t|^{3/2}}\,\mathrm{d}s\right|^p \mathrm{d}x\,\mathrm{d}t\right)^{1/p} \\
&\lesssim 2^{-3k} \left(\int_{A_k} \left|\int_{t_i-1}^{t_i+1} f(x,s)\eta_j(x,s)\,\mathrm{d}s\right|^p \mathrm{d}x\,\mathrm{d}t\right)^{1/p}.
\end{aligned}$$

Now $|\int_A f| \le \int_A |f| \le |A|^{1/p'} \left(\int_A |f|^p\right)^{1/p}$ and so $|\int_A f|^p \le |A|^{p/p'} \int_A |f|^p$. Hence using Minkowski's integral inequality, Fubini and that $p \ge 1$

$$\begin{aligned}
\|D^t_{1/2}(f\eta_j)\|_{L^p(A_k)} &\lesssim 2^{-3k} \left(\int_{A_k} 2^{p-1} \int_{t_i-1}^{t_i+1} |f(x,s)\eta_j(x,s)|^p \,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t\right)^{1/p} \\
&\lesssim 2^{-3k} \left(\int_{A_k} \int_{t_i-1}^{t_i+1} |f(x,s)\eta_j(x,s)|^p \,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t\right)^{1/p} \\
&\lesssim 2^{-3k} \left(\int_{t_i-2^{2k+1}}^{t_i+2^{2k+1}} \int_{\mathbb{R}^n} |f(x,s)\eta_j(x,s)|^p \,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t\right)^{1/p} \\
&\lesssim 2^{-3k+2k/p} \|f\|_{L^p(B_j)} \lesssim 2^{-k} \|f\|_{L^p(B_j)}.
\end{aligned}$$

Then summing over A_k from $k=1$ we obtain

$$\|D^t_{1/2}(f\eta_j)\|_{L^p(\mathbb{R}^n \setminus 2B_j)} \lesssim \|f\|_{L^p(B_j)}.$$

Finally summing in j and using the bounded overlap property of B_j gives

$$\sum_j \|D^t_{1/2}(f\eta_j)\|^p_{L^p(\mathbb{R}^n\setminus 2B_j)} \lesssim M\|f\|^p_{L^p(\mathbb{R}^n)}. \tag{3.2.23}$$

Step 3.b: $D^t_{1/2}(f\eta_j)(x,t)$ when $(x,t)\in 2B_j$.
We want to show the following bound

$$\|D^t_{1/2}(f\eta_j)\|_{L^p(2B_j)} \lesssim \|D^t_{1/2}f\|_{L^p(4B_j)} + \|f\|_{L^p(4B_j)},$$

then we would again sum over j and use the bounded overlapping property of B_j to obtain

$$\sum_j \|D^t_{1/2}(f\eta_j)\|^p_{L^p(2B_j)} \lesssim M^4\|D^t_{1/2}f\|^p_{L^p(\mathbb{R}^n)} + M^4\|f\|^p_{L^p(\mathbb{R}^n)}.$$

We can rewrite $D^t_{1/2}(f\eta_j)(x,t)$ as

$$D^t_{1/2}(f\eta_j)(x,t) = \int_{\mathbb{R}} \frac{f(x,s)\left(\eta_j(x,s)-\eta_j(x,t)\right)}{|s-t|^{3/2}}\,\mathrm{d}s + \eta_j(x,t)D^t_{1/2}f(x,t). \tag{3.2.24}$$

Due to the bounded overlap, the $\eta_j(x,t)D^t_{1/2}f(x,t)$ term behaves nicely when we take L^p norms, and sum so we only need to concentrate on the left hand term. We split this into 2 parts

$$\begin{aligned}\int_{\mathbb{R}} \frac{f(x,s)\left(\eta_j(x,s)-\eta_j(x,t)\right)}{|s-t|^{3/2}}\,\mathrm{d}s &= \int_{|t-s|<4} \frac{f(x,s)\left(\eta_j(x,s)-\eta_j(x,t)\right)}{|s-t|^{3/2}}\,\mathrm{d}s \\ &\quad + \int_{|t-s|>4} \frac{f(x,s)\left(\eta_j(x,s)-\eta_j(x,t)\right)}{|s-t|^{3/2}}\,\mathrm{d}s \\ &= T_I f + T_{II} f\end{aligned}$$

and want to show

$$\begin{aligned}\|T_I f\|^p_{L^p(2B_j)} &\lesssim \|f\|^p_{L^p(4B_j)}, \\ \|T_{II} f\|^p_{L^p(2B_j)} &\lesssim \|f\|^p_{L^p(4B_j)}.\end{aligned}$$

Step 3.b.i: Boundedness of T_I.
In this step we use an interpolation argument[1] by showing

$$\begin{aligned}T_I &: L^1(4B_j)\to L^1(2B_j) \\ T_I &: L^\infty(4B_j)\to L^\infty(2B_j).\end{aligned}$$

Since the η_j's are uniformly Lipschitz

$$T_I f(x,t) = \int_{|t-s|<4} \frac{f(x,s)\left(\eta_j(x,s)-\eta_j(x,t)\right)}{|s-t|^{3/2}}\,\mathrm{d}s \lesssim \int_{|t-s|<4} \frac{|f(x,s)|}{|s-t|^{1/2}}\,\mathrm{d}s.$$

First start with L^1 boundedness and using Fubini:

$$\begin{aligned}\|T_I f\|_{L^1(2B_j)} &\lesssim \int_{2B_j}\int_{|t-s|<4} \frac{|f(x,s)|}{|s-t|^{1/2}}\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}x \lesssim \int_{4B_j}\int_{|t-s|<4} \frac{|f(x,s)|}{|s-t|^{1/2}}\,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}x \\ &\lesssim \int_{4B_j} |f(x,s)| \int_{|t-s|<4} \frac{1}{|s-t|^{1/2}}\,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}x \\ &\lesssim \int_{4B_j} |f(x,s)|\,\mathrm{d}s\,\mathrm{d}x = \|f\|_{L^1(4B_j)}.\end{aligned}$$

We tackle the L^∞ case by noting that if $(x,t)\in 2B_j$ ans since $|t-s|<4$ then $(x,s)\in 4B_j$.

[1] For an introduction to interpolation see [BL76].

Hence

$$\|T_I f\|_{L^\infty(2B_j)} \lesssim \int_{|t-s|<4} \frac{|f(x,s)|}{|s-t|^{1/2}}\,\mathrm{d}s \lesssim \|f\|_{L^\infty(4B_j)}.$$

By interpolation ([BL76]) we get the desired result.

Step 3.b.ii: Boundedness of T_{II} for $(x,t) \in 2B_j$.

$$\begin{aligned} T_{II} f(x,t) &= \int_{|t-s|>4} \frac{f(x,s)\,(\eta_j(x,s) - \eta_j(x,t))}{|s-t|^{3/2}}\,\mathrm{d}s \\ &= \int_{|t-s|>4} \frac{-f(x,s)\eta_j(x,t)}{|s-t|^{3/2}}\,\mathrm{d}s \end{aligned}$$

since $(x,t) \in 2B_j$ and $|t-s| > 4$ then $(x,s) \notin 2B_j$. We apply Hölder's inequality

$$\begin{aligned} \|T_{II} f\|^p_{L^p(2B_j)} &= \int_{2B_j} \left| \int_{|t-s|>4} \frac{f(x,s)\eta_j(x,s)}{|s-t|^{3/2}}\,\mathrm{d}s \right|^p \mathrm{d}t\,\mathrm{d}x \\ &\le \int\limits_{2B_j} \left(\int\limits_{|t-s|>4} \frac{|f(x,s)|^p \eta_j(x,s)^p}{|s-t|^{3/2}}\,\mathrm{d}s \right) \left(\int\limits_{|t-s|>4} \frac{1}{|s-t|^{3p'/2}}\,\mathrm{d}s \right)^{p/p'} \mathrm{d}t\,\mathrm{d}x \\ &\lesssim \int_{2B_j} \int_{|t-s|>4} |f(x,s)|^p \eta_j(x,s)^p\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}x \\ &\lesssim \int_{t_i-4}^{t_i+4} \int_{\mathbb{R}^n} |f(x,s)|^p \eta_j(x,s)^p\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t \\ &\lesssim \|f\|^p_{L^p(B_j)}. \end{aligned}$$

This finishes the proof of the L^p boundedness of T_I and T_{II}.

Combining all the estimates above in step 3.b we have

$$D^t_{1/2}(f\eta_j)(x,t) = T_I f + T_{II} f + \eta_j(x,t) D^t_{1/2} f(x,t)$$

and

$$\begin{aligned} \|D^t_{1/2}(f\eta_j)\|^p_{L^p(2B_j)} &\le \|T_I f\|^p_{L^p(2B_j)} + \|T_{II} f\|^p_{L^p(2B_j)} + \|\eta_j D^t_{1/2} f\|^p_{L^p(2B_j)} \\ &\lesssim \|f\|^p_{L^p(4B_j)} + \|f\|^p_{L^p(B_j)} + \|D^t_{1/2} f\|^p_{L^p(B_j)}. \end{aligned}$$

Now summing over j and using the bounded overlap ($4B_j$ is covered by a bounded number of B_j depending only on M, d and n)

$$\sum_j \|D^t_{1/2}(f\eta_j)\|^p_{L^p(2B_j)} \lesssim \|f\|^p_{L^p(\mathbb{R}^n)} + \|D^t_{1/2} f\|^p_{L^p(\mathbb{R}^n)}. \tag{3.2.25}$$

Therefore combining (3.2.23) and (3.2.25) with step 3.a we have shown (3.2.21)

$$\sum_j \|D^t_{1/2}(f\eta_j)\|^p_{L^p(\mathbb{R}^n)} \lesssim \|f\|^p_{L^p(\mathbb{R}^n)} + \|D^t_{1/2} f\|^p_{L^p(\mathbb{R}^n)}.$$

Step 3.c: Finally we are left with proving (3.2.22).

$$\|D^t_{1/2} f\|^p_{L^p(\mathbb{R}^n)} \lesssim \sum_j \|D^t_{1/2}(f\eta_j)\|^p_{L^p(\mathbb{R}^n)} + \|f\|^p_{L^p(\mathbb{R}^n)}.$$

From (3.2.24) we have

$$D^t_{1/2}f(x,t) = \sum_j D^t_{1/2}(f\eta_j)(x,t) - \sum_j \int_{\mathbb{R}} \frac{f(x,s)\,(\eta_j(x,s)-\eta_j(x,t))}{|s-t|^{3/2}}\,\mathrm{d}s.$$

So applying norms, the triangle inequality and using the convexity of $t \mapsto t^p$ for $p \geq 1$ gives

$$\|D^t_{1/2}f\|^p_{L^p(\mathbb{R}^n)} \lesssim \sum_j \|D^t_{1/2}(f\eta_j)\|^p_{L^p(\mathbb{R}^n)} + \sum_j \left\| \int_{\mathbb{R}} \frac{f(x,s)\,(\eta_j(x,s)-\eta_j(x,t))}{|s-t|^{3/2}}\,\mathrm{d}s \right\|^p_{L^p_{x,t}(\mathbb{R}^n)}.$$

We want to show we can control the second term by $\|f\|_{L^p(\mathbb{R}^n)}$. The argument is similar to that given in steps 3.a and 3.b. We first look at the norm away from $2B_j$ and by (3.2.23) in step 3.a we have

$$\sum_j \left\| \int_{\mathbb{R}} \frac{f(x,s)\,(\eta_j(x,s)-\eta_j(x,t))}{|s-t|^{3/2}}\,\mathrm{d}s \right\|^p_{L^p_{x,t}(\mathbb{R}^n\setminus 2B_j)} \lesssim M\|f\|^p_{L^p(\mathbb{R}^n)}.$$

When $(x,t) \in 2B_j$ then

$$\int_{\mathbb{R}} \frac{f(x,s)\,(\eta_j(x,s)-\eta_j(x,t))}{|s-t|^{3/2}}\,\mathrm{d}s = T_I f + T_{II} f.$$

Therefore by the bounds in step 3.b and similar to (3.2.25) we have that

$$\sum_j \left\| \int_{\mathbb{R}} \frac{f(x,s)\,(\eta_j(x,s)-\eta_j(x,t))}{|s-t|^{3/2}}\,\mathrm{d}s \right\|^p_{L^p_{x,t}(2B_j)} \lesssim \|f\|^p_{L^p(\mathbb{R}^n)}.$$

Combining these two bounds gives us what we require, that

$$\|D^t_{1/2}f\|^p_{L^p} \lesssim \sum_j \|D^t_{1/2}(f\eta_j)\|^p_{L^p} + \|f\|^p_{L^p} \lesssim \sum_j \|D^t_{1/2}(f\eta_j)\|^p_{L^p} + \sum_j \|f\eta_j\|^p_{L^p}.$$

Hence we have shown (3.2.22) and proved (3.2.20). ☐

3.3 Proof of Theorem 3.1.1

Before we begin the proof of theorem 3.1.1 we state a result that we use.

Lemma 3.3.1 (Poincaré type inequality, [Zie89, Corollary 4.5.3])**.** *If $u \in W^{1,p}(E)$ and $p > 1$ then*

$$\|u\|_{L^{p^*}(E)} \leq C(B_{1,p}(N))^{-1/p}\|Du\|_{L^p(E)}, \tag{3.3.1}$$

where $B_{\alpha,p}(E)$ is the Bessel capacity of the set E [2]*; N is the set where u vanishes, i.e. $N = \{x : u(x) = 0\}$; and $p^* = \frac{np}{n-p}$ if $p < n$, $1 \leq p^* < \infty$ if $p = n$, and $p^* = \infty$ if $p > n$.*

We apply this to the case where the set E is a time slice of $T(\Delta_r)$.

Corollary 3.3.2. *Let $u \in W^{1,p}(T(\Delta_r)|_{t'})$, where $u = 0$ on $\Delta_r|_{t'}$ for a fixed time t'. Let $p > 1$ then there is a constant C independent of r such that*

$$\|u\|_{L^p_x\left(T(\Delta_r)|_{t'}\right)} \leq Cr\|\nabla u\|_{L^p_x\left(T(\Delta_r)|_{t'}\right)}. \tag{3.3.2}$$

Proof. The case for $r = 1$ follows from the positivity of $B_{\alpha,p}\left(\Delta_1|_{t'}\right)$ [Zie89, §2.6], lemma 3.3.1, and Hölder's inequality. For a general r apply the substitution $v(x) := u(rx)$ then $v \in W^{1,p}\left(T(\Delta_r)|_{t'}\right)$. Applying the $r = 1$ case and a change of variables gives the general result. ☐

[2] See [Zie89] for a definition of Bessel capacity.

The proof of theorem 3.1.1 uses some of the ideas from Kenig and Pipher's [KP93] proof in the elliptic setting. However, due to the time irreversibility of parabolic equations we do not have the comparison principle, the Carleson estimate [CFMS81, Theorem 1.1] or Harnack's principle that they used. Also, the non-commutativity of taking the adjoint and the pullback mapping introduce additional difficulties. Instead, we get around these problems using the Green's function in a more nuanced way; the maximum principle; a different Carleson type estimate (lemma 2.2.8); approaching some estimates from an integral instead of a pointwise point of view; and using the Hardy-Littlewood maximal function.

Proof of theorem 3.1.1. Assume that $(R)_p$ holds for (3.1.1) and let ω^* be the parabolic measure associated to the adjoint equation (3.1.2). By theorem 2.4.29 to show that $(D^*)_{p'}$ holds we need to show that $\omega^* \ll \sigma$, where σ is the measure on $\partial\Omega$ in definition 2.1.7, and ω^* belongs to the reverse Hölder class $B_p(\mathrm{d}\sigma)$, see definition 2.4.27.

We first prove the reverse Hölder inequality (2.4.17) for surface balls that fit inside a cylinder $2\mathbb{Z}_j$ and then use a covering argument to show that (2.4.17) holds for all balls with the correct scaling. Note that since (2.1.10) holds in $2\mathbb{Z}_j$ so we can replace σ by $\mathscr{H}^n$.

Step 1: Preliminaries.

Let Δ_d be a surface ball on $\partial\Omega$, with d from definition 2.1.3, then Δ_d lies completely inside an ℓ-cylinder $2\mathbb{Z}_j$. After we apply ρ_j, the pullback transformation in definition 2.1.5, Δ_d becomes a surface ball Δ_d on ∂U, where U is the upper half space[3]. Let $\Delta_r(y,a) \subset \Delta_d \subset U$ be a surface ball such that $4r < d$. Note if we omit the point that Δ_r is centred at then it will be centred at $(y,a) \in \partial U$ (or $(Y,s) \in \partial\Omega$).

As in [KP93], we define a non-negative C_0^∞ function f on ∂U as follows: $f = 0$ on Δ_r, $f = 1$ on $\Delta_{3r} \setminus \Delta_{2r}$ and $f = 0$ on $\partial U \setminus \Delta_{4r}$ with $|\nabla_T f| \lesssim 1/r$ and $|\partial_t f| \lesssim 1/r^2$. Here we note that $\Delta_{4r} \subset \Delta_d$. Using theorem 3.2.3 and interpolation we have

$$\begin{aligned} \int_{\partial U} |\nabla_T f|^p \,\mathrm{d}\mathscr{H}^n &\lesssim r^{n+1-p}, \\ \int_{\partial U} |\mathbb{D}_n f|^p \,\mathrm{d}\mathscr{H}^n &\lesssim \|D^t_{1/2} f\|^p_{L^p} + \|\nabla f\|^p_{L^p} \lesssim \|\partial_t f\|^{p/2}_{L^p} \|f\|^{p/2}_{L^p} + \|\nabla f\|^p_{L^p} \\ &\lesssim r^{n+1-p}. \end{aligned} \tag{3.3.3}$$

By Sobolev embedding, since $f \in C_0^\infty(\Delta_d)$, for a fixed time t

$$\int_{\mathbb{R}^{n-1}} |f(x,t)|^p \,\mathrm{d}x \lesssim \int_{\mathbb{R}^{n-1}} |\nabla_T f(x,t)|^p \,\mathrm{d}x. \tag{3.3.4}$$

Here and in the following estimate the implied constant depends on d. Integrating (3.3.4) in time gives

$$\int_{\partial U} |f|^p \,\mathrm{d}\mathscr{H}^n \lesssim \int_{\partial U} |\nabla_T f|^p \,\mathrm{d}\mathscr{H}^n \lesssim r^{n+1-p}. \tag{3.3.5}$$

It follows that $f^u = f \circ \rho_j^{-1}$ is $\Delta_d(Y,a)$ supported boundary data on $\partial\Omega$ with $L^p_{1,1/2}(\partial\Omega, d\sigma)$ norm comparable to $r^{(n+1)/p-1}$.

Since we assume $(R)_p$ solvability for the equation (3.1.4) let u be the solution of (3.1.4) in Ω with boundary data f^u. We then have for u the following estimate for $1 < q \leq 2$

$$\|\widetilde{N}_q(\nabla u)\|_{L^p(\partial\Omega)} \lesssim \|\widetilde{N}_2(\nabla u)\|_{L^p(\partial\Omega)} \lesssim \|f\|_{L^p_{1,1/2}} \lesssim r^{(n+1)/p-1}. \tag{3.3.6}$$

Let $s \ll r$ (we are going to take limit $s \to 0+$) and let $(P,b) \in \partial\Omega$ be a point on the boundary such that $\Delta_{10s}(P,b) \subset \Delta_r$.

Step 2: Equivalence between the Green's function and the parabolic measure.

[3] For simplicity we have ignored the Lipschitz deformation of the ball since this only modifies the estimate by a uniform constant depending on ℓ. We may shrink the affected balls as necessary to overcome this.

We now have three surface balls $\Delta_s \subset \Delta_r \subset \Delta_d$. Let V_s^-, V_r^- and V_d^- be their respective corkscrew points shifted backwards in time from their centres by $100s^2$, $100r^2$ and $100d^2$ respectively. Therefore, by applying lemmas 2.3.6 and 2.3.7 we have

$$\frac{\omega^{*V_d^-}(\Delta_s(P,b))}{\omega^{*V_d^-}(\Delta_r)} \sim \frac{s^n}{r^n}\frac{G^*(V_d^-,V_s^-)}{G^*(V_d^-,V_r^-)} = \frac{s^n}{r^n}\frac{G(V_s^-,V_d^-)}{G(V_r^-,V_d^-)}. \tag{3.3.7}$$

Step 3: Controlling Green's function by the solution u.

Recall $G(\cdot,V_d^-)$ is a solution to (3.1.1). For this step in this proof we want to show that (3.3.7) can be uniformly controlled by $u(V_s^-)s^n/r^n$ for all $s \ll r$. To this end, we show that $G(X,t,V_d^-) \lesssim u(X,t)G(V_r^-,V_d^-)$ on the boundary of $T(\Delta_{5r/2})$ and then apply the maximum principle (lemma 2.2.5) to show that $G(X,t,V_d^-) \lesssim u(X,t)G(V_r^-,V_d^-)$ for $(X,t) \in T(\Delta_{5r/2})$.

On $\Delta_{5r/2}$ we have that $0 = G(\cdot,V_d^-) \leq u(\cdot)G(V_r^-,V_d^-)$ so we are left to show that $G^{V_d^-}(X) \lesssim G^{V_d^-}(V_r^-)$ and $u \sim 1$ on $\partial T(\Delta_{5r/2}) \setminus \partial\Omega$, i.e. the interior piece of $\partial T(\Delta_{5r/2})$.

Step 3.a: $G(X,t,V_d^-) \lesssim G(V_r^-,V_d^-)$ on $\partial T(\Delta_{5r/2}) \setminus \partial\Omega$.

Here we use that $T(\Delta_{5r/2})$ is later than V_r^- in time, i.e. $T(\Delta_{5r/2}) \subset \{t > a-(9r)^2\}$. For points (X,t) in $\partial T(\Delta_{5r/2})$ away from $\partial\Omega$ we can just apply the Harnack inequality (lemma 2.2.4) to conclude that $G(X,t,V_d^-) \lesssim G(V_r^-,V_d^-)$. For points (X,t) near $\partial\Omega$ we can apply the Carleson type estimate (lemma 2.2.8), to obtain $G(X,t,V_d^-) \lesssim G(V^-(\Delta_r(Z,\tau)),V_d^-)$, where (Z,τ) is any point in $\Delta_{5r/2}$. Since V_r^- is at an earlier time than $V^-(\Delta_r(Z,\tau))$, we can again apply the Harnack inequality to obtain $G(X,t,V_d^-) \lesssim G(V_r^-,V_d^-)$ for $(X,t) \in T(\Delta_r(Z,\tau))$. From this the claim follows.

Step 3.b: $u \sim 1$ on $\partial T(\Delta_{5r/2}) \setminus \partial\Omega$.

As before, near to $\partial\Omega$ applying the Carleson type estimate (lemma 2.2.8) to $1-u$ gives us that $u(X,t) \sim 1$ for $(X,t) \in \Psi_{r/4}(Z,\tau)$, where $(Z,\tau) \in \partial\Delta_{5r/2}$. Away from $\partial\Omega$ we use Harnack's inequality (lemma 2.2.4) to conclude that $u \sim 1$ at a later time when $\partial T(\Delta_{5r/2}) \cap \partial\Omega$.

Step 3.c: Applying the maximum principle.

By applying the maximum principle, we have that $G(X,t,V_d^-) \lesssim u(X,t)G(V_r^-,V_d^-)$ for $(X,t) \in T(\Delta_{5r/2})$ and since $V_s^- \in T(\Delta_{5r/2})$ we may conclude that $G(V_s^-,V_d^-) \lesssim u(V_s^-)G(V_r^-,V_d^-)$.

We have now proved

$$\frac{\omega^{*V_d^-}(\Delta_s(P,b))}{\omega^{*V_d^-}(\Delta_r)} \lesssim \frac{s^n}{r^n}u(V_s^-). \tag{3.3.8}$$

Step 4: Applying the Poincaré type inequality to the spacial variables (corollary 3.3.2) at a fixed time $t = t'$ we have for $q > 1$

$$\left(\fint_{T(\Delta_s(P,b))|_{t'}} |u(X,t)|^q \,\mathrm{d}X\right)^{1/q} \lesssim s\left(\fint_{T(\Delta_s(P,b))|_{t'}} |\nabla u(X,t)|^q \,\mathrm{d}X\right)^{1/q}.$$

Then averaging in time over $(b-s^2, b+s^2)$ gives

$$\left(\fint_{T(\Delta_s(P,b))} |u(X,t)|^q \,\mathrm{d}X\,\mathrm{d}t\right)^{1/q} \lesssim s\left(\fint_{T(\Delta_s(P,b))} |\nabla u(x,t)|^q \,\mathrm{d}X\,\mathrm{d}t\right)^{1/q}. \tag{3.3.9}$$

By applying the Harnack inequality to $u(V_s^-)$, we can estimate the value of u at this point by the infimum of u over the ball $Q_{s/8}\left(V_s^- + (0, s^2/4^2)\right)$ (the centre of this ball is V_s^- shifted by $s/4$ in time). It follows that

$$u(V_s^-) \lesssim \inf_{Q_{s/8}(V_s+(0,s^2/4^2))} u \lesssim \left(\fint_{Q_{s/2}(V_s)} |u(X,t)|^q \,\mathrm{d}X \,\mathrm{d}t \right)^{1/q}$$
$$\lesssim s \left(\fint_{T(\Delta_{12s}(P,b))} |\nabla u(X,t)|^q \,\mathrm{d}X \,\mathrm{d}t \right)^{1/q}.$$

Therefore

$$\frac{\omega^{*V_d^-}(\Delta_s(P,b))}{\omega^{*V_d^-}(\Delta_r)} \lesssim \frac{s^n}{r^n} u(V_s^-) \lesssim \frac{s^{n+1}}{r^n} \left(\fint_{T(\Delta_{12s}(P,b))} |\nabla u(X,t)|^q \,\mathrm{d}X \,\mathrm{d}t \right)^{1/q}. \tag{3.3.10}$$

Step 5: We would like to bound (3.3.10) by $\widetilde{N}_2(\nabla u)(P,b)$, the L^2 based non-tangential maximal function. This is easy to do in the elliptic setting but it is not clear whether it is possible to do in our setting due to the time irreversibility of the parabolic PDE. Instead, we claim that we have the following bound

$$\left(\fint_{T(\Delta_{12s}(P,b))} |\nabla u(X,t)|^q \,\mathrm{d}X \,\mathrm{d}t \right)^{1/q} \lesssim \left(M\left(\widetilde{N}_q(\nabla u)^q \right)(P,b) \right)^{1/q}, \tag{3.3.11}$$

where M is the Hardy-Littlewood maximal function defined using parabolic surface balls.

Let T_i be the subset of $T(\Delta_{12s}(P,b))$ at a distance $2^{-i}s$ from the boundary. More formally $T_i := \{(x_0,x,t) : (x,t) \in \Delta_{12s}(P,b) \text{ and } 2^{-i}12s < \operatorname{dist}(X,t) < 2^{-i+1}12s\}$. Then

$$\int_{T(\Delta_{12s}(P,b))} |\nabla u(X,t)|^q \,\mathrm{d}X \,\mathrm{d}t = \sum_{i=1}^{\infty} \int_{T_i} |\nabla u(X,t)|^q \,\mathrm{d}X \,\mathrm{d}t.$$

Using $|T(\Delta_{12s}(P,b))| \sim 2^i |T_i|$ we have that

$$\fint_{T(\Delta_{12s}(P,b))} |\nabla u(X,t)|^q \,\mathrm{d}X \,\mathrm{d}t \lesssim \sum_{i=1}^{\infty} 2^{-i} \fint_{T_i} |\nabla u(X,t)|^q \,\mathrm{d}X \,\mathrm{d}t. \tag{3.3.12}$$

We now show that on each piece

$$\fint_{T_i} |\nabla u(X,t)|^q \,\mathrm{d}X \,\mathrm{d}t \lesssim \fint_{\Delta_{12s}(P,b)} \widetilde{N}_q(\nabla u)(X,t)^q \,\mathrm{d}X \,\mathrm{d}t.$$

If the distance from the boundary of (X,t) is in the middle of the slice of T_i, that is $\operatorname{dist}((X,t),\partial\Omega) = \frac{3}{2}2^{-i}12s$, then the ball $B_{2^{-i}12s}(X,t)$ is one of those considered in the supremum of $\widetilde{N}_q(\nabla u)(X,t)$. Therefore

$$\fint_{B_{2^{-i}12s}(X,t)} |\nabla u|^q \leq \widetilde{N}_q(\nabla u)(X,t)^q$$

and hence

$$\int_{B_{2^{-i}12s}(X,t)} |\nabla u|^q \leq \left(2^{-i}12s\right)^{n+2} \widetilde{N}_q(\nabla u)(X,t)^q.$$

By integrating this over the surface ball $\Delta_{12s}(P,b)$

$$\left(2^{-i+1}12s\right)^{n+1} \int_{T_i} |\nabla u(X,t)|^q \,\mathrm{d}X \,\mathrm{d}t \lesssim \int_{\Delta_{12s}(P,b)} \left(\int_{B_{2^{-i}12s}(X,t)} |\nabla u|^q \right) \mathrm{d}X \,\mathrm{d}t$$
$$\leq \left(2^{-i}12s\right)^{n+2} \int_{\Delta_{12s}(P,b)} \widetilde{N}_q(\nabla u)(X,t)^q \,\mathrm{d}X \,\mathrm{d}t.$$

Consequently

$$\int_{T_i} |\nabla u(X,t)|^q \,\mathrm{d}X\,\mathrm{d}t \lesssim 2^{-i}s \int_{\Delta_{12s}(P,b)} \widetilde{N}_q(\nabla u)(X,t)^q \,\mathrm{d}X\,\mathrm{d}t.$$

Using the relationship $|T_i| \sim 2^{-i}s|\Delta_{12s}(P,b)|$

$$\fint_{T_i} |\nabla u(X,t)|^q \,\mathrm{d}X\,\mathrm{d}t \lesssim \fint_{\Delta_{12s}(P,b)} \widetilde{N}_q(\nabla u)(X,t)^q \,\mathrm{d}X\,\mathrm{d}t \lesssim M\left(\widetilde{N}_q(\nabla u)^q\right)(P,b)$$

and therefore using (3.3.12) we have proved (3.3.11).

Combining (3.3.10) with (3.3.11) we have

$$\frac{\omega^{*V_d^-}(\Delta_s(P,b))}{\sigma(\Delta_s(P,b))} \lesssim \frac{\omega^{*V_d^-}(\Delta_r)}{r^n}\left(M\left(\widetilde{N}_q(\nabla u)^q\right)(P,b)\right)^{1/q}, \tag{3.3.13}$$

where as before $s < r/10$ and (P,b) is such that $\Delta_{10s}(P,b) \subset \Delta_r$. In particular this estimate holds for all $(P,b) \in \Delta_{r/2}$.

Step 6: The B_p condition.

To show the property $(D^*)_{p'}$ we need to show that $K^{V_d^-} = \mathrm{d}\omega^{*V_d^-}/\,\mathrm{d}\sigma$ belongs to the reverse Hölder class $B_p(\mathrm{d}\sigma)$, c.f. definition 2.4.27. To do this we take the same approach as [KP93]. Let

$$h^{V_d^-}(P,b) := \sup_{s\in(0,r/10)} \frac{\omega^{*V_d^-}(\Delta_s(P,b))}{\sigma(\Delta_s(P,b))},$$

then $K^{V_d^-}(P,b) \le h^{V_d^-}(P,b)$ for $(P,b) \in \Delta_{r/2}$. Since $(M(|f|^q))^{1/q}$ is L^p bounded for $1 < q < p$ and $\widetilde{N}_q(f) \le \widetilde{N}_2(f)$ for $0 < q \le 2$ we choose $q \in (1, \min\{2,p\})$ to conclude

$$\|K^{V_d^-}\|_{L^p(\mathrm{d}\sigma)} \le \|h^{V_d^-}\|_{L^p(\mathrm{d}\sigma)} \lesssim \frac{\omega^{*V_d^-}(\Delta_r)}{r^n}\left\|\widetilde{N}_q(\nabla u)\right\|_{L^p(\mathrm{d}\sigma)} < \infty. \tag{3.3.14}$$

Therefore $K^{V_d^-}, h^{V_d^-} \in L^p(\mathrm{d}\sigma)$ and so $\omega^{*V_d^-} \ll \sigma$.

Using (3.3.14), (3.3.6) and the doubling property of ω^* (lemma 2.3.4) the weight $K^{V_d^-}$ satisfies the B_p condition (2.4.17) for the ball $\Delta_{r/2}$

$$\begin{aligned}\left(\frac{1}{\sigma\left(\Delta_{r/2}\right)}\int_{\Delta_{r/2}}\left(K^{V_d^-}\right)^p \mathrm{d}\sigma\right)^{1/p} &\lesssim \frac{\omega^{*V_d^-}(\Delta_r)}{r^n}\left(\frac{1}{r^{n+1}}r^{n+1-p}\right)^{1/p}\\ &\lesssim \frac{\omega^{*V_d^-}(\Delta_r)}{\sigma(\Delta_r)} \lesssim \frac{\omega^{*V_d^-}(\Delta_{r/2})}{\sigma(\Delta_{r/2})}.\end{aligned} \tag{3.3.15}$$

By considering different balls Δ_r we can conclude that the above inequality holds for any surface ball $\Delta_r \subset \Delta_d$ with $4r \le d$.

One can then use lemmas 2.3.4 and 2.3.5 to see the above reverse Hölder inequality holds for all balls up to size d. Let $r \le d$ then we may cover Δ_r by up to N balls $\{\Delta_{r_j}\}$ with $\Delta_{4r_j} \subset \Delta_j$. Δ_j is a surface ball of radius d and $\Delta_j \subset 2\mathbb{Z}_j$. The reason for this argument is that Δ_j may belong to different ℓ-cylinders $\mathbb{Z}_j$. Let V_j be the corresponding (backward in time) corkscrew point of each Δ_j and V_d be a corkscrew point of Δ_d earlier in time than all V_j's. We apply (3.3.15) to each of these balls then

$$\left(\frac{1}{\sigma\left(\Delta_{r_j}\right)}\int_{\Delta_{r_j}}\left(K^{V_j}\right)^p \mathrm{d}\sigma\right)^{1/p} \lesssim \frac{\omega^{*V_j}\left(\Delta_{r_j}\right)}{\sigma\left(\Delta_{r_j}\right)}.$$

By changing the corkscrew point (lemma 2.3.5) and the doubling property of ω^* (lemma 2.3.4)

$$\left(\frac{1}{\sigma\left(\Delta_r\right)}\int_{\Delta_{r_j}}\left(K^{V_d}\right)^p \,\mathrm{d}\sigma\right)^{1/p}\lesssim\frac{\omega^{*V_d}\left(\Delta_r\right)}{\sigma\left(\Delta_r\right)}$$

and since there are at most N balls we obtain

$$\left(\frac{1}{\sigma\left(\Delta_r\right)}\int_{\Delta_r}\left(K^{V_d}\right)^p \,\mathrm{d}\sigma\right)^{1/p}\lesssim\frac{\omega^{*V_d}\left(\Delta_r\right)}{\sigma\left(\Delta_r\right)}.$$

It follows that the $L^{p'}$ Dirichlet problem for the adjoint PDE (3.1.5) is solvable in Ω. $\square$

Chapter 4

The L^p Dirichlet Problem

This chapter studies time-varying domains and the L^p Dirichlet problem when the coefficients satisfy a Carleson condition. We start by giving a brief survey of the known L^p Dirichlet results in time-varying domains. One of the largest sections in this chapter, section 4.2, focuses on investigating the Lewis-Murray condition. We begin by motivating this condition from the perspective of layer potentials and review the known equivalent conditions for $D^t_{1/2}\phi \in \text{BMO}$ if ϕ is $\text{Lip}(1, 1/2)$. Following this we find three new equivalent conditions (one already known but with a questionable proof) with equivalence of norms to $\|\mathbb{D}\phi\|_*$. These results and the proofs behind them (with inspiration from [Str80]) may be of independent interest especially in the setting of parabolic PDE in time-varying domains or parabolic uniform rectifiability. We further prove that one of these conditions is localisable. Finally we are able to state our definition of admissible and VMO-type domains, and we compare them to the definition of Lewis-Murray cylinders. We define our pullback mapping from Ω to the upper half space and after a modification of a proof from [HL96] show how under this transformation a Carleson condition of our coefficients is preserved, c.f. (1.0.7).

After showing that the basic inequalities from section 2.2 still hold in our domain if we introduce a small drift term, we move onto some delicate arguments showing that the p-adapted square and p-adapted area functions are well defined (the integrals are not a priori locally integrable). Further we show a weighted Caccioppoli inequality for the second gradient which shows that we can bound the p-adapted area function by the p-adapted square function. Subsequently we can begin to prove the solvability of the L^p Dirichlet problem, theorem 4.1.6. We do this via the standard non-tangential maximal and square function arguments and, although there are a few new terms to deal with when we integrate by parts, there are no major new ideas needed here.

The results from this chapter appear in a shortened form in [DDH18].

4.1 Introduction

Recall that we are studying solutions to the following parabolic problem

$$\begin{cases} u_t = \operatorname{div}(A\nabla u) + B \cdot \nabla u & \text{in } \Omega \subset \mathbb{R}^{n+1}, \\ u = f & \text{on } \partial\Omega. \end{cases} \tag{4.1.1}$$

In this section we review some of the important recent work studying the L^p Dirichlet problem and then state our main theorem. Later in section 4.2 we give formal definitions of the domains we call *Lewis-Murray cylinders* (definition 4.2.3), which occur in [DH18; DPP17]; and our *admissible domains* and VMO-*type domains* are defined in definition 4.2.20 in section 4.2.3, from [DDH18]. From the definitions and theorem 4.2.13, every admissible domain is a Lewis-Murray cylinder and vice versa but the admissible domain definition is easier to verify. However, the main advantage of admissible domains is that they have a much more nuanced approach to the norms of the graphs allowing us to give stronger solvability results for a given domain. This permits us to make L^p Dirichlet solvability statements for a larger class of domains. Intuitively a *Lewis-Murray cylinder* is a $\mathrm{Lip}(1,1/2)$ cylinder given by global functions ϕ_j which also satisfy the Lewis-Murray condition; an *admissible domain* is given by local functions ϕ_j which satisfy the Lewis-Murray condition and are allowed to have a large $\mathrm{Lip}(1,1/2)$ norm; and a VMO-*type domain* is a specific case of an admissible domain which has vanishing BMO norms.

Theorem 4.1.1 ([HL96])**.** *Let Ω be a graph domain given by a $\mathrm{Lip}(1,1/2)$ function ϕ with norm ℓ and let $D^t_{1/2}\phi \in \mathrm{BMO}(\mathbb{R}^n)$. Given $1 < p < \infty$ if $\|D^t_{1/2}\phi\|_*$ and ℓ are sufficiently small then (D_p) holds for the heat equation in Ω. Furthermore, when $p = 2$ we can remove the smallness assumption on ℓ.*

The following theorem is the local parabolic analogue of [KKPT00] and a converse of lemma 2.4.31 and proposition 2.4.32.

Theorem 4.1.2 ([Riv03])**.** *Let Ω be a graph domain given by a compact $\mathrm{Lip}(1,1/2)$ function ϕ with norm ℓ and $D^t_{1/2}\phi \in \mathrm{BMO}(\mathbb{R}^n)$. Let u be a bounded solution to (4.1.1) with no drift term. If for every surface cube Δ_r we have the following comparability between non-tangential maximal and square functions*

$$\begin{aligned}\int_{\Delta_r} Nu^2 \,\mathrm{d}\sigma &\lesssim \int_{2\Delta_r} Su^2 \,\mathrm{d}\sigma + r^{n+1}|u(V(\Delta_r))|^2,\\ \int_{\Delta_r} Su^2 \,\mathrm{d}\sigma &\lesssim \int_{2\Delta_r} Nu^2 \,\mathrm{d}\sigma\end{aligned} \tag{4.1.2}$$

then $\omega \in A_\infty(\mathrm{d}\sigma)$.

The next A_∞ results come from [DPP17] and are the parabolic analogue of [KKPT15]. They are inspired by the classical results of [Fef71; FS72] which say $f \in \mathrm{BMO}$ is equivalent to $\mathrm{d}\mu = x_0|\nabla u|^2 \,\mathrm{d}x \,\mathrm{d}x_0$ is a Carleson measure in the upper half space (if we assume the BMO growth condition (2.5.4)).

Theorem 4.1.3 ([DPP17])**.** *Let Ω be a Lewis-Murray cylinder from definition 4.2.3 with character (ℓ, N, d). If the parabolic measure $\omega \in A_\infty(\mathrm{d}\sigma)$ then for all continuous functions $f \in C_0(\partial\Omega)$ the solution u to (4.1.1) (without a drift term) satisfies the estimate*

$$\sup_{\Delta \subset \partial\Omega} \frac{1}{\sigma(\Delta)} \int_{T(\Delta)} |\nabla u|^2 \delta \,\mathrm{d}X \,\mathrm{d}t \lesssim \|f\|_*^2.$$

Rivera-Noriega [Riv12] has a similar result but requires much stronger assumptions upon the domain — star-like; satisfying a stronger Lewis-Murray type condition (c.f. (4.2.27) and remark 4.2.11); and needs control over $\sup_{T(\Delta)} |u|^2$.

Theorem 4.1.4 ([DPP17])**.** *Let Ω be a Lewis-Murray cylinder from definition 4.2.3 with character (ℓ, N, d). If for all continuous functions $f \in C_0(\partial\Omega)$ the solution u to (4.1.1) (without a drift term) satisfies the estimate*

$$\sup_{\Delta \subset \partial\Omega} \frac{1}{\sigma(\Delta)} \int_{T(\Delta)} |\nabla u|^2 \delta \,\mathrm{d}X \,\mathrm{d}t \lesssim \|f\|^2_{L^\infty(\partial\Omega,\, \mathrm{d}\sigma)}$$

then the parabolic measure $\omega \in A_\infty(\mathrm{d}\sigma)$.

The difference between the two theorems above is that to establish A_∞ we only need to test with the bigger L^∞ norm on the right hand side instead of the BMO norm.

The following theorem is similar to our main result for this chapter with the main differences being that theorem 4.1.6 establishes (D_p) for the full range of p and allows a more general class of domains — we do not need a small Lipschitz norm and our ϕ_j's are not assumed to be compactly supported (or global functions).

Theorem 4.1.5 ([DH18])**.** *Let Ω be a Lewis-Murray cylinder from definition 4.2.3 with character (ℓ, N, d). Consider any $2 \le p \le \infty$ and assume that either:*

(1)

$$\mathrm{d}\mu = \left[\delta(X,t)^{-1}\sup_{i,j}\left(\operatorname*{osc}_{B_{\delta(X,t)/2}(X,t)} a_{ij}\right)^2 + \delta(X,t)\sup_{B_{\delta(X,t)/2}(X,t)}|B|^2\right]\mathrm{d}X\,\mathrm{d}t \tag{4.1.3}$$

is the density of a Carleson measure on Ω *with Carleson norm* $\|\mu\|_C$*.*

(2) Or

$$\mathrm{d}\mu = \left(\delta(X,t)|\nabla A|^2 + \delta(X,t)^3|\partial_t A|^2 + \delta(X,t)|B|^2\right)\,\mathrm{d}X\,\mathrm{d}t \tag{4.1.4}$$

is the density of a Carleson measure on Ω *with Carleson norm* $\|\mu\|_C$ *and*

$$\delta(X,t)|\nabla A| + \delta(X,t)^2|\partial_t A| + \delta(X,t)|B| \leq \|\mu\|_C^{1/2} \tag{4.1.5}$$

is the density of a Carleson measure on Ω*. Then there exists* $\varepsilon > 0$ *such that if for some* $r_0 > 0$ *we have* $\max(\ell^2, \|\mu\|_{C,r_0}) < \varepsilon$ *then the* L^p *Dirichlet problem is solvable for all* $2 \leq p \leq \infty$*.*

Further parabolic results For further results see the following papers [LS88; LM92; LM95; HL96; HL01; DPP17; Riv14]. When A is independent in the x_0 direction see these papers [Nys16; AEN16; Nys17] which are related to the parabolic version of the Kato square root problem. For how this relates to parabolic uniform rectifiability see [HLN03; HLN04; NS17] and references therein.

We ready to state our main result; some notions used here are defined in detail in chapter 2 and section 4.2.

Theorem 4.1.6. *Let* Ω *be an admissible domain as in definition 4.2.20 with character* (ℓ, η, N, d)*, let* A *be bounded and elliptic* (1.0.2)*, and* B *be measurable. Consider any* $1 < p \leq \infty$ *and assume that either:*

(1)

$$\mathrm{d}\mu = \left[\delta(X,t)^{-1}\sup_{i,j}\left(\operatorname*{osc}_{B_{\delta(X,t)/2}(X,t)} a_{ij}\right)^2 + \delta(X,t)\sup_{B_{\delta(X,t)/2}(X,t)}|B|^2\right]\mathrm{d}X\,\mathrm{d}t \tag{4.1.6}$$

is the density of a Carleson measure on Ω *with Carleson norm* $\|\mu\|_C$*.*

(2) Or

$$\mathrm{d}\mu = \left(\delta(X,t)|\nabla A|^2 + \delta(X,t)^3|\partial_t A|^2 + \delta(X,t)|B|^2\right)\,\mathrm{d}X\,\mathrm{d}t \tag{4.1.7}$$

is the density of a Carleson measure on Ω *with Carleson norm* $\|\mu\|_C$ *and*

$$\delta(X,t)|\nabla A| + \delta(X,t)^2|\partial_t A| + \delta(X,t)|B| \leq \|\mu\|_C^{1/2}. \tag{4.1.8}$$

Then there exists $K = K(\lambda, \Lambda, \ell, n, p) > 0$ *such that if for some* $r_0 > 0$

$$\max\{\eta, \|\mu\|_{C,r_0}\} < K$$

then the L^p *Dirichlet boundary value problem* (4.1.1) *is solvable. Moreover, the following estimate holds for all continuous boundary data* $f \in C_0(\partial\Omega)$

$$\|N(u)\|_{L^p(\partial\Omega,\,\mathrm{d}\sigma)} \lesssim \|f\|_{L^p(\partial\Omega,\,\mathrm{d}\sigma)},$$

where the implied constant depends only on the operator, n*,* p *and character* (ℓ, η, N, d)*.*

Corollary 4.1.7. *In particular, if* Ω *is of VMO-type (*η *in the character* (ℓ, η, N, d) *can be taken arbitrary small), and the Carleson measure* μ *from theorem 4.1.6 is a vanishing Carleson measure then the* L^p *Dirichlet boundary value problem is solvable for all* $1 < p \leq \infty$*.*

4.2 Parabolic Time-varying Domains

In this section we define a class of time-varying domains whose boundaries are given locally as functions $\phi(x,t)$, Lipschitz in the spatial variable and satisfying the Lewis-Murray condition in the time variable. At each time $\tau \in \mathbb{R}$ the set of points in Ω with fixed time $t = \tau$, that is $\Omega_\tau = \Omega \cap \{t = \tau\}$, is still a non-empty bounded Lipschitz domain in $\mathbb{R}^n$. We start with a discussion of the Lewis-Murray condition and in section 4.2.1 give motivation for choosing this condition. In section 4.2.2 we give a summary and clarification of the Lewis-Murray condition in the literature, and introduce three new equivalent definitions. We also correct a wrong definition found in the literature and set ourselves up for a localisation result. We continue examination of the Lewis-Murray condition in section 4.2.3 and localise one of the equivalent definitions of the condition. We prove that this is a good localisation and we can extend local functions that satisfy this to global functions which satisfy the usual Lewis-Murray condition. Finally in this subsection we are able to state our definition of an admissible parabolic domain used in theorem 4.1.6. Last of all, in section 4.2.4 we study the pullback transformation, and see how this relates to our admissible parabolic domains and the Carleson condition on the coefficients.

We remind the reader that the Lewis-Murray condition imposed that a pointwise half derivative in time of $\phi(x,t)$ belongs to parabolic BMO, $D^t_{1/2}\phi \in \mathrm{BMO}$.

4.2.1 Motivation

It had been thought that the correct parabolic analogue of Lipschitz domains were $\mathrm{Lip}(1, 1/2)$ domains, due to the parabolic scaling. However, Kaufman and Wu [KW88] produced the following counterexample which showed that the natural surface measure σ failed to be in A_∞ for the heat equation in a drastic way.

Theorem 4.2.1 ([KW88])**.** *One may explicitly construct a graph domain $\Omega = \{(x_0,t) \in \mathbb{R}\times\mathbb{R} : x_0 > \phi(t)\}$ with $\phi \in \mathrm{Lip}(1/2)$ such that the parabolic measure ω and the adjoint parabolic measure ω^* to the heat equation are concentrated on two disjoint sets whose projections onto the t-axis have Hausdorff dimensions strictly less than 1.*

Kaufman and Wu [KW88] built their example domain to be a fractal constructed from a simple periodic curve. The first four iterations of the construction of this domain are shown in figure 4.1 on p. 49.

Why the Lewis-Murray condition is the parabolic analogue of Lipschitz domains

Lewis and Murray [LM95] showed their result using the method of layer potentials (which are singular integral operators). In the elliptic setting the layer potential method comes from the boundedness of the Cauchy integral on Lipschitz curves[1]; see [Tay00, Chapter 4] for an overview of layer potentials on Lipschitz domains for elliptic equations. In the proof of the boundedness of the Cauchy integral on Lipschitz curves via Calderón commutators we can show the $L^2 \to L^2$ bound using the $T(1)$ theorem. Here the operator is split into an infinite sum of commutators with bounds that depend on the first commutator. The bound of the first Calderòn commutator is exactly

$$\left\| \left[\sqrt{\Delta}, \phi\right] \right\|_{L^2\to L^2} \sim \|\nabla\phi\|_{L^\infty},$$

the Lipschitz norm of the graph ϕ.

In the parabolic setting Hofmann [Hof95] showed the analogue of this

$$\left\| \left[\sqrt{\Delta - \frac{\partial}{\partial t}}, \phi\right] \right\|_{L^2\to L^2} \sim \|\nabla\phi\|_{L^\infty} + \|D^t_{1/2}\phi\|_*. \tag{4.2.1}$$

Therefore when the parabolic layer potential operator is split into Calderòn-type commutators [LM92; LM95; HL96] the $L^2 \to L^2$ boundedness of the first commutator in (4.2.1) prescribes the regularity required of the domain.

[1]For a proof of the boundedness of the Cauchy integral on Lipschitz curves via Calderón commutators and the $T(1)$ theorem see [Duo01].

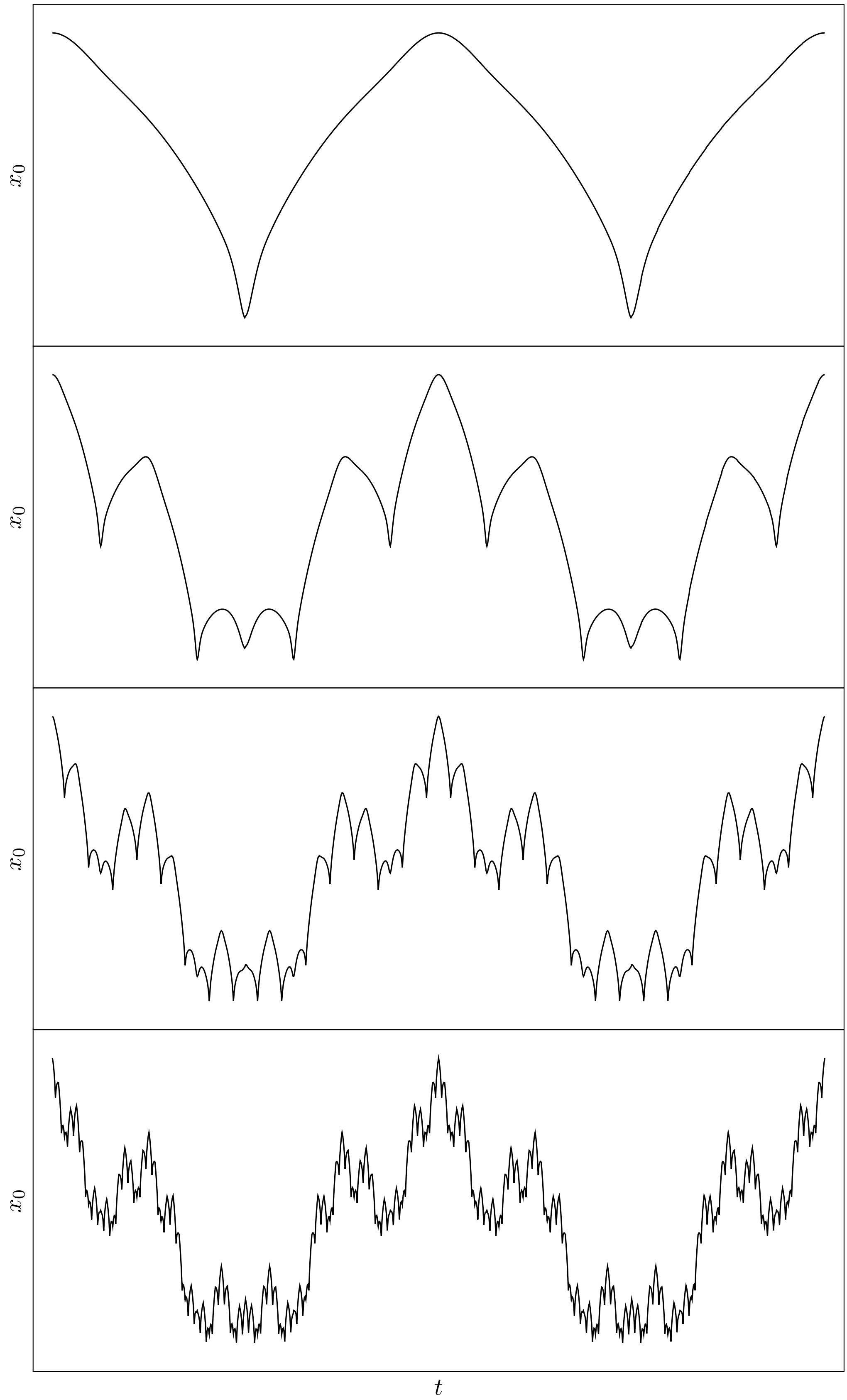

Figure 4.1: The first four iterations of the counterexample domain built by Kaufman and Wu [KW88] for theorem 4.2.1.

Furthermore, there is a sharp restriction on the size of $\|D^t_{1/2}\phi\|_*$ (and no restriction on the size of $\|\nabla\phi\|_{L^\infty}$).

Theorem 4.2.2 ([HL96]). *Given $1 < p < \infty$ there exists a constant $C(p)$ and a graph domain given by a $\mathrm{Lip}(1, 1/2)$ function ϕ such that $\|D^t_{1/2}\phi\|_* \leq C(p)$ but the L^p Dirichlet and L^p regularity problems for the heat equation are not solvable.*

Definition 4.2.3 (Lewis-Murray cylinder). $\Omega \subset \mathbb{R}^n \times \mathbb{R}$ *is a Lewis-Murray cylinder with character (ℓ, N, d) if for any time $\tau \in \mathbb{R}$ there are at most N ℓ-cylinders $\{\mathbb{Z}_j\}_{j=1}^N$ of diameter d satisfying the following conditions:*

(1) $\partial\Omega \cap \{|t - \tau| \leq d^2\} = \bigcup_j (\mathbb{Z}_j \cap \partial\Omega)$.

(2) In the coordinate system (x_0, x, t) of the ℓ-cylinder $\mathbb{Z}_j$

$$\mathbb{Z}_j \cap \Omega \supset \left\{(x_0, x, t) \in \Omega : |x| < d, |t| < d^2, \delta(x_0, x, t) \leq d/2\right\}.$$

(3) $8\mathbb{Z}_j \cap \partial\Omega$ *is the graph* $\{x_0 = \phi_j(x, t)\}$ *of a global function* $\phi_j : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}$ *such that*

$$|\phi_j(x, t) - \phi_j(y, s)| \leq \ell\left(|x - y| + |t - s|^{1/2}\right) \quad \text{and} \quad \phi_j(0, 0) = 0. \tag{4.2.2}$$

(4)

$$\|D^t_{1/2}\phi_j\|_{\mathrm{BMO}(\mathbb{R}^{n-1}\times\mathbb{R})} \leq \ell. \tag{4.2.3}$$

This defines the same class of domains as in [DH18; DPP17] but if one compares the definitions to those in the papers cite it can seen that here we have been explicit about the support of ϕ.

Remark 4.2.4. By a result of Strichartz [Str80] extended to the parabolic case by Hofmann [Hof95], $|\phi_j(x, t) - \phi_j(y, t)| \leq \ell|x - y|$ and $\|D^t_{1/2}\phi_j\|_* \leq \eta$ imply that ϕ_j is $\mathrm{Lip}(1, 1/2)$ and satisfies (4.2.2) with a comparable constant

$$|\phi_j(x, t) - \phi_j(x, s)| \lesssim (\ell + \eta)|t - s|^{1/2}.$$

4.2.2 The Lewis-Murray Condition: Review and New Results

There are a few different ways by which one can define half derivatives and BMO-Sobolev spaces and there are also some erroneous results in the literature which we correct here. To bring clarity, we start by discussing the various definitions in the global setting of a graph domain $\Omega = \{(x_0, x, t) : x_0 > \phi(x, t)\}$, where $\phi : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}$. We review the known results from [HL96] in theorem 4.2.5, introduce three new equivalent definitions of the Lewis-Murray condition in theorem 4.2.7 and then in remark 4.2.11 we discuss an incorrect statement found in [Riv03]. We follow the standard notation of [HL96].

Recall from section 3.2 if $g \in C_0^\infty(\mathbb{R})$ and $0 < \alpha < 2$ then the *one-dimensional fractional differentiation operators* D_α are defined on the Fourier side by

$$\widehat{D_\alpha g}(\tau) = |\tau|^\alpha \hat{g}(\tau).$$

If $0 < \alpha < 1$ then by standard results

$$D_\alpha g(t) = c \int_{\mathbb{R}} \frac{g(t) - g(s)}{|t - s|^{1+\alpha}} \, \mathrm{d}s.$$

Therefore, we define the *pointwise half derivative in time* of $\phi : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}$ to be

$$D^t_{1/2}\phi(x, t) = c_n \int_{\mathbb{R}} \frac{\phi(x, s) - \phi(x, t)}{|s - t|^{3/2}} \, \mathrm{d}s, \tag{4.2.4}$$

for a properly chosen constant c_n, c.f. [HL96].

However, this definition ignores the spatial coordinates. Remember we instead followed [FR67] and we defined the *parabolic half derivative in time* of $\phi : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}$ to be

$$\widehat{\mathbb{D}_n\phi}(\xi, \tau) = \frac{\tau}{\|(\xi, \tau)\|}\hat{\phi}(\xi, \tau), \tag{4.2.5}$$

where ξ and τ denote the spatial and temporal variables on the Fourier side respectively. In addition we define the *parabolic derivative* (in space and time) of $\phi : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}$ to be

$$\widehat{\mathbb{D}\phi}(\xi, \tau) = \|(\xi, \tau)\|\hat{\phi}(\xi, \tau). \tag{4.2.6}$$

Note that $\mathbb{D}^{-1}$ is the parabolic Riesz potential. One can also represent $\mathbb{D}$ as

$$\mathbb{D} = \sum_{j=1}^{n} R_j \mathbb{D}_j, \tag{4.2.7}$$

where $\mathbb{D}_j = \partial_j$ for $1 \leq j \leq n-1$, $\mathbb{D}_n$ is defined above, and R_j are the parabolic Riesz transforms introduced in example 2.5.28. The parabolic Riesz transforms are defined on the Fourier side as

$$\begin{aligned} \widehat{R_j}(\xi, \tau) &= \frac{i\xi_j}{\|(\xi, \tau)\|} \quad \text{for } 1 \leq j \leq n-1 \text{ and} \\ \widehat{R_n}(\xi, \tau) &= \frac{\tau}{\|(\xi, \tau)\|^2}. \end{aligned} \tag{4.2.8}$$

From corollary 2.5.27 each R_j defines a bounded operator on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, is bounded on $\mathrm{BMO}(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$, and preserves the class H^1_{00}, c.f. [Pee66; FR66; FR67; HL96].

The *Lewis-Murray condition* on the domain Ω, for which they proved A_∞ [LM92; LM95], is $\phi \in \mathrm{Lip}(1, 1/2)$ and $\|D^t_{1/2}\phi\|_* \leq \eta$; note this BMO norm is taken over $\mathbb{R}^{n-1} \times \mathbb{R}$.

It is worth remarking that neither the operators $D^t_{1/2}$, $\mathbb{D}_n$ or $\mathbb{D}$ easily lend themselves to being localised to a function $\phi : Q_d \to \mathbb{R}$ due to their non-local natures. However, our goal in this chapter is provide a theory where the domain is locally given by graphs which satisfy the Lewis-Murray condition. The parabolic nature of the PDE (especially time irreversibility and exponential decay of solutions with vanishing boundary data) suggests we should only expect to need local conditions on the functions describing the boundary. To this end we state the following theorems where we show six equivalent statements to the Lewis-Murray condition for a global function $\phi : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}$. Furthermore, the final conditions admit themselves to being both localised easily and amiable to extension; see theorem 4.2.13 later for details on an extension.

We return to this localisation thought in section 4.2.3 but first we motivate and discuss the Lewis-Murray condition.

The equivalence of conditions (1) and (2) below is shown in [HL96] with an equivalence of norms in the small and large sense, see [HL96, (2.10) and Theorem 7.4] for precise details.

Theorem 4.2.5. *Let $\phi : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}$ and $\phi \in \mathrm{Lip}(1, 1/2)$ then the following conditions are equivalent:*

(1) $D^t_{1/2}\phi \in \mathrm{BMO}(\mathbb{R}^n)$.

(2) $\mathbb{D}_n\phi \in \mathrm{BMO}(\mathbb{R}^n)$.

(3) $\mathbb{D}\phi \in \mathrm{BMO}(\mathbb{R}^n)$.

Furthermore $\mathbb{D}_n\phi = R_n\mathbb{D}\phi$ and so $\|\mathbb{D}_n\phi\|_* \lesssim \|\mathbb{D}\phi\|_*$.

We now extend this theorem by adding three more equivalent statements in theorem 4.2.7. To motivate condition (6) of theorem 4.2.7 below we first recall a characterisation of BMO in terms of means on adjacent cubes from [Str80, p. 546]. Let $M(f, Q) = \frac{1}{|Q|}\int_Q f$ denote the average of f over a cube Q, and let $\tilde{Q}_\rho(x)$ be the cube of radius ρ with x in the upper right corner.

Lemma 4.2.6 ([Str80]). *$f \in \mathrm{BMO}(\mathbb{R}^n)$ is equivalent to*

$$\sup_{Q_r} \sum_{k=1}^{n} \frac{1}{|Q_r|} \int_{Q_r} \int_0^r \left| M(f, \tilde{Q}_\rho(x)) - M(f, \tilde{Q}_\rho(x - \rho e_k)) \right|^2 \frac{\mathrm{d}\rho}{\rho} \,\mathrm{d}x = B < \infty, \tag{4.2.9}$$

where e_k are the usual unit vectors in $\mathbb{R}^n$, and $\|f\|_*^2 \sim B$.

The equivalence of conditions (3) and (4) in the theorem below is a generalisation of [Str80] to the parabolic setting that is stated in [Riv03], c.f. [FS72; CT75; CT77]. We have some question-marks over the proof given in [Riv03]; however the argument we give for condition (5) also works for condition (4) and hence the claim in [Riv03] is correct.

Theorem 4.2.7. *Let $\phi : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}$ and $\phi \in \mathrm{Lip}(1, 1/2)$ then the following conditions are equivalent:*

(3) $\mathbb{D}\phi \in \mathrm{BMO}(\mathbb{R}^n)$.

(4)

$$\sup_{Q_r} \frac{1}{|Q_r|} \int_{Q_r} \int_{\|(y,s)\| \le r} \frac{|\phi(x+y, t+s) - 2\phi(x,t) + \phi(x-y, t-s)|^2}{\|(y,s)\|^{n+3}} \,\mathrm{d}y \,\mathrm{d}s \,\mathrm{d}x \,\mathrm{d}t = B_{(4)} < \infty. \tag{4.2.10}$$

(5) (a)

$$\sup_{Q_r} \frac{1}{|Q_r|} \int_{Q_r} \int_{|y|<r} \frac{|\phi(x+y, t) - 2\phi(x,t) + \phi(x-y, t)|^2}{|y|^{n+1}} \,\mathrm{d}y \,\mathrm{d}x \,\mathrm{d}t = B_{(5.a)} < \infty, \tag{4.2.11}$$

(b)

$$\sup_{Q_r = J_r \times I_r} \frac{1}{|Q_r|} \int_{Q_r} \int_{I_r} \frac{|\phi(x,t) - \phi(x,s)|^2}{|t-s|^2} \,\mathrm{d}s \,\mathrm{d}t \,\mathrm{d}x = B_{(5.b)} < \infty. \tag{4.2.12}$$

(6) Let $u = (u', u_n) \in \mathbb{S}^{n-1}$ and let e_n be the unit vector in the time direction. For $k = 1, \dots, n-1$ let

$$A_k = \int_0^1 \rho u' \cdot \left(M\left(\nabla\phi, \tilde{Q}_\rho(x + \lambda\rho u', t)\right) - M\left(\nabla\phi, \tilde{Q}_\rho(x + \lambda\rho u' - \rho e_k, t)\right)\right) \mathrm{d}\lambda,$$

$$A_n = \int_0^1 \rho u' \cdot \left(M\left(\nabla\phi, \tilde{Q}_\rho(x + \lambda\rho u', t)\right) - M\left(\nabla\phi, \tilde{Q}_\rho(x + \lambda\rho u', t - \rho^2)\right)\right) \mathrm{d}\lambda.$$

(a)

$$\sup_{Q_r} \sum_{k=1}^{n} \frac{1}{|Q_r|} \int_{Q_r} \int_{u \in \mathbb{S}^{n-1}} \int_0^r \frac{|A_k|^2}{\rho^3} \,\mathrm{d}\rho \,\mathrm{d}u \,\mathrm{d}x \,\mathrm{d}t = B_{(6.a)} < \infty, \tag{4.2.13}$$

(b)

$$\sup_{Q_r = J_r \times I_r} \frac{1}{|Q_r|} \int_{Q_r} \int_{I_r} \frac{|\phi(x,t) - \phi(x,s)|^2}{|t-s|^2} \,\mathrm{d}s \,\mathrm{d}t \,\mathrm{d}x = B_{(6.b)} < \infty. \tag{4.2.12}$$

Furthermore we have equivalence of the norms

$$\|\mathbb{D}\phi\|_*^2 \sim B_{(4)} \sim B_{(5.a)} + B_{(5.b)} \sim B_{(6.a)} + B_{(6.b)}. \tag{4.2.14}$$

Remark 4.2.8. Condition (6.a) doesn't immediately look too similar to its supposed motivation, (4.2.9) in lemma 4.2.6. However, if we move back into Cartesian coordinates and undo the mean value theorem then we obtain something much more similar to a combination of (4.2.9) and an endpoint version of [Str80, (3.1)]. The reason why we can obtain the endpoint, whereas [Str80, (3.1)] can only be used for a fractional derivative strictly smaller than 1, is due to extra integrability and cancellation coming from (4.2.16) and (4.2.17). Consider

$$
\begin{aligned}
A'_k &= M\left(\phi, \tilde{Q}_{\|(y,s)\|}(x+y,t)\right) - M\left(\phi, \tilde{Q}_{\|(y,s)\|}(x,t)\right) \\
&\quad - M\left(\phi, \tilde{Q}_{\|(y,s)\|}(x+y-\|(y,s)\|e_k,t)\right) + M\left(\phi, \tilde{Q}_{\|(y,s)\|}(x-\|(y,s)\|e_k,t)\right), \\
A'_n &= M\left(\phi, \tilde{Q}_{\|(y,s)\|}(x+y,t)\right) - M\left(\phi, \tilde{Q}_{\|(y,s)\|}(x,t)\right) \\
&\quad - M\left(\phi, \tilde{Q}_{\|(y,s)\|}(x+y,t-\|(y,s)\|^2)\right) + M\left(\phi, \tilde{Q}_{\|(y,s)\|}(x,t-\|(y,s)\|^2)\right)
\end{aligned}
$$

then condition (6.a) is equivalent to

$$
\sup_{Q_r} \sum_{k=1}^{n} \frac{1}{|Q_r|} \int_{Q_r} \int_{\|(y,s)\|<r} \frac{|A'_k|^2}{\|(y,s)\|^{n+3}} \, \mathrm{d}y \, \mathrm{d}s \, \mathrm{d}x \, \mathrm{d}t = \tilde{B}_{(6.a)} < \infty. \tag{4.2.15}
$$

Proof of theorem 4.2.7. We begin by proving the equivalence of conditions (3) and (6) using ideas from [Str80] and write $F = \mathbb{D}\phi$ where F is a tempered distribution. Let

$$
\varphi^k = \chi_{\tilde{Q}_1(0,0)} - \chi_{\tilde{Q}_1(e_k)} \tag{4.2.16}
$$

then for $1 \le k \le n-1$

$$
\begin{aligned}
\widehat{\varphi^k}(\xi,\tau) &= \frac{2\sin^2(\xi_k/2)}{\xi_k} \frac{1-e^{-i\tau}}{i\tau} \prod_{j\neq k}^{n-1} \frac{1-e^{-i\xi_j}}{i\xi_j}, \\
\widehat{\varphi^n}(\xi,\tau) &= \frac{2\sin^2(\tau/2)}{\tau} \prod_{j=1}^{n-1} \frac{1-e^{-i\xi_j}}{i\xi_j},
\end{aligned} \tag{4.2.17}
$$

with $\widehat{\varphi^k}(\xi,\tau) \sim \xi_k$ for small ξ_k and $1 \le k \le n-1$. We let

$$
\widehat{\psi^u} = \frac{e^{i(\xi,0)\cdot u} - 1}{\|(\xi,\tau)\|}
$$

and denote by $\psi^u_\rho(x,t)$ the usual parabolic dilation by ρ, that is

$$
\psi^u_\rho(x,t) = \rho^{-(n+1)} \psi^u(x/\rho, t/\rho^2).
$$

It is worth noting that $(\varphi^k * \psi^u)_\rho = \varphi^k_\rho * \psi^u_\rho$. Therefore we may rewrite condition (6.a), by remark 4.2.8, as

$$
\sup_{Q_r} \sum_{k=1}^{n-1} \frac{1}{|Q_r|} \int_{Q_r} \int_{u\in\mathbb{S}^{n-1}} \int_0^r \left(\psi^u_\rho * \varphi^k_\rho * F\right)^2 \frac{\mathrm{d}\rho}{\rho} \, \mathrm{d}u \, \mathrm{d}x \, \mathrm{d}t \sim B_{(6.a)}. \tag{4.2.18}
$$

Similarly if we let

$$
\widehat{\psi^u_n} = \frac{e^{i(0,\tau)\cdot u} - 1}{\|(\xi,\tau)\|} \tag{4.2.19}
$$

then we may rewrite condition (6.b) as

$$
\sup_{Q_r} \frac{1}{|Q_r|} \int_{Q_r} \int_{u\in\mathbb{S}^{n-1}} \int_0^r \left(\psi^u_{n,\rho} * F\right)^2 \frac{\mathrm{d}\rho}{\rho} \, \mathrm{d}u \, \mathrm{d}x \, \mathrm{d}t \sim B_{(6.b)}. \tag{4.2.20}
$$

The functions $\varphi^k * \psi^u$ and ψ_n^u all satisfy the following conditions for some $\varepsilon_i > 0$

$$\begin{aligned}
&\int \psi \,\mathrm{d}x \,\mathrm{d}t = 0, \\
&|\psi(x,t)| \lesssim \|(x,t)\|^{-n-1-\varepsilon_1} \text{ for } \|(x,t)\| \geq a > 0, \\
&|\widehat{\psi}(\xi,\tau)| \lesssim \|(\xi,\tau)\|^{\varepsilon_2} \text{ for } \|(\xi,\tau)\| \leq 1, \\
&|\widehat{\psi}(\xi,\tau)| \lesssim \|(\xi,\tau)\|^{-\varepsilon_3} \text{ for } \|(\xi,\tau)\| \geq 1.
\end{aligned} \tag{4.2.21}$$

Therefore if $\mathbb{D}\phi = F \in \mathrm{BMO}(\mathbb{R}^n)$ then $B_{(6.a)} \lesssim \|\mathbb{D}\phi\|_*^2$ and $B_{(6.b)} \lesssim \|\mathbb{D}\phi\|_*^2$ by theorem 2.5.14 and remark 2.5.15; this shows condition (3) implies condition (6).

For the converse we proceed via an analogue of the proof of [Str80, Theorem 2.6]. Consider

$$\hat{\theta}(\xi,\tau) = \|(\xi,\tau)\|\hat{\zeta}(\xi,\tau),$$

where $\zeta \in C_0^\infty(\mathbb{R})$. As before let H^1_{00} be the dense subclass of Schwartz H^1 functions, see definition 2.5.9 and [Ste70, p. 225]. Via an analogue of [FS72, Theorem 3; Str80, Lemma 2.3] by assuming conditions (6.a) and (6.b) if $g \in H^1_{00}(\mathbb{R}^n)$ then for each $1 \leq k \leq n-1$

$$\left| \int_{\mathbb{S}^{n-1}} \int_0^\infty \iint\limits_{\mathbb{R}^{n-1}\times\mathbb{R}} \psi_\rho^u * \varphi_\rho^k * F(x,t)\theta_\rho * g(x,t) \,\mathrm{d}x \,\mathrm{d}t \frac{\mathrm{d}\rho}{\rho} \,\mathrm{d}u \right| \lesssim B_{(6.a)}^{1/2} \|g\|_{H^1}, \tag{4.2.22}$$

and

$$\left| \int_{\mathbb{S}^{n-1}} \int_0^\infty \iint\limits_{\mathbb{R}^{n-1}\times\mathbb{R}} \psi_{n,\rho}^u * F(x,t)\theta_\rho * g(x,t) \,\mathrm{d}x \,\mathrm{d}t \frac{\mathrm{d}\rho}{\rho} \,\mathrm{d}u \right| \lesssim B_{(6.b)}^{1/2} \|g\|_{H^1}. \tag{4.2.23}$$

For $1 \leq k \leq n-1$ let

$$\begin{aligned}
m_k(\xi,\tau) &= \int_{\mathbb{S}^{n-1}} \int_0^\infty \overline{\hat{\psi}^u\left(-\rho\xi, -\rho^2\tau\right) \hat{\varphi}^k\left(-\rho\xi,-\rho^2\tau\right)} \|(\xi,\tau)\|\zeta(\rho\|(\xi,\tau)\|) \,\mathrm{d}\rho \,\mathrm{d}u, \\
m_n(\xi,\tau) &= \int_{\mathbb{S}^{n-1}} \int_0^\infty \overline{\hat{\psi}_n^u\left(-\rho\xi,-\rho^2\tau\right)} \|(\xi,\tau)\|\zeta(\rho\|(\xi,\tau)\|) \,\mathrm{d}\rho \,\mathrm{d}u.
\end{aligned} \tag{4.2.24}$$

All of these functions m_k, for $1 \leq k \leq n$, are homogeneous of degree zero and smooth away from the origin. The associated Fourier multipliers M_k are Caldorón-Zygmund operators that preserve the class H^1_{00} and are bounded on H^1, by theorem 2.5.26 and corollary 2.5.27.

The non-degeneracy condition from [CT75] (c.f. definition 2.5.13) on the family of functions $\{m_k\}_{k=1}^n$ holds — that is the property that $\sum_k |m_k(r\xi, r^2\tau)|^2$ does not vanish identically in r for $(\xi,\tau) \neq (0,0)$. Therefore by [CT75; CT77] we can find smooth, homogeneous of degree zero functions $u_{k,j}(\xi,\tau)$ and positive numbers r_j such that for all $(\xi,\tau) \neq (0,0)$

$$\sum_{k=1}^n \sum_{j=1}^{j_0} m_{k,r_j}(\xi,\tau) u_{k,j}(\xi,\tau) = 1. \tag{4.2.25}$$

Here m_{k,r_j} are as m_k but with $r_j\rho$ replacing ρ in the arguments of $\hat{\psi}^u$, $\hat{\varphi}^k$ and $\hat{\psi}_n^u$ in (4.2.24) (but not ζ). That is

$$\begin{aligned}
m_{k,r_j}(\xi,\tau) &= \int\limits_{\mathbb{S}^{n-1}} \int_0^\infty \overline{\hat{\psi}^u\left(-r_j\rho\xi, -(r_j\rho)^2\tau\right) \hat{\varphi}^k\left(-r_j\rho\xi, -(r_j\rho)^2\tau\right)} \|(\xi,\tau)\|\zeta(\rho\|(\xi,\tau)\|) \,\mathrm{d}\rho \,\mathrm{d}u, \\
m_{n,r_j}(\xi,\tau) &= \int\limits_{\mathbb{S}^{n-1}} \int_0^\infty \overline{\hat{\psi}_n^u\left(-r_j\rho\xi, -(r_j\rho)^2\tau\right)} \|(\xi,\tau)\|\zeta(\rho\|(\xi,\tau)\|) \,\mathrm{d}\rho \,\mathrm{d}u.
\end{aligned}$$

Let $M_{k,j}$ and $U_{k,j}$ be the associated Fourier multiplier operators to their respective multipliers m_{k,r_j} and $u_{k,j}$ then $\sum\sum M_{k,j}U_{k,j}g = g$ for all $g \in H^1_{00}$. By [FS72, Theorem 3; Str80, Lemma

2.4] there exists $h_{k,j} \in \mathrm{BMO}(\mathbb{R}^n)$ such that $\|h_{k,j}\|_*^2 \lesssim B_{(6.a)}$ or $B_{(6.b)}$, and $(h_{k,j}, g) = (F, M_{k,j}g)$ for all $g \in H^1_{00}$. If we replace g by $U_{j,k}g \in H^1_{00}$ in the previous identity and sum over j and k we obtain $(h, g) = (F, g)$ for all $g \in H^1_{00}$ where $h = \sum_{k,j} U^*_{k,j} h_{k,j}$; furthermore by the BMO condition on $h_{k,j}$, $\|h\|_*^2 \lesssim B_{(6.a)} + B_{(6.b)}$. The identity (4.2.25) does not need to hold at the origin therefore $\hat{h} - \hat{F}$ may be supported at the origin and hence $F = h + p$ where p is a polynomial. Due to the assumption $\phi \in \mathrm{Lip}(1, 1/2)$ in can be clearly seen that F must be a tempered distribution. Hence as in [Str80] we may conclude $F = h \in \mathrm{BMO}(\mathbb{R}^n)$, modulo a constant. This implies equivalence of conditions (3) and (6).

Similarly we may prove the equivalence of conditions (4) and (5) to condition (3). The changes needed are outlined below.

Condition (5) $\iff$ condition (3) In this instance we replace the convolutions $\varphi^k * \psi^u$ by

$$\hat{\psi}^u_1(\xi, \tau) = \frac{e^{i(\xi,0)\cdot u} - 2 - e^{-i(\xi,0)\cdot u}}{\|(\xi, \tau)\|},$$

which corresponds to condition (5.a), and we keep the convolution ψ^u_n as it is in (4.2.19). The same proof then goes through to give that condition (5) holds if and only if condition (3) holds with equivalent norms as in (4.2.14).

Condition (4) $\iff$ condition (3) This case is stated in [Riv03, Proposition 3.2]. Again the proof proceeds as above but with only one convolution needed

$$\hat{\psi}^u(\xi, \tau) = \frac{e^{i(\xi,\tau)\cdot u} - 2 - e^{-i(\xi,\tau)\cdot u}}{\|(\xi, \tau)\|}. \qquad \square$$

Proposition 4.2.9. $\nabla\phi \in \mathrm{BMO}(\mathbb{R}^n)$ *implies condition (6.a) and* $\nabla\phi(\cdot, t) \in \mathrm{BMO}(\mathbb{R}^{n-1})$ *uniform a.e. in time implies condition (5.a), with the constants* $B_{(5.a)}$ *and* $B_{(6.a)}$ *controlled by the appropriate* $\|\nabla\phi\|_*^2$ *norms. Here* $\mathrm{BMO}(\mathbb{R}^{n-1})$ *denotes the* BMO *norm in the spatial variables only.*

Proof. The statement $\nabla\phi \in \mathrm{BMO}(\mathbb{R}^{n-1})$ implies condition (5.a) follows from [Str80, Theorem 3.3]. In order to establish the first claim for the ease of notation let us fix Q_r and k such that $1 \le k \le n-1$. Since $|u'| \le 1$ after changing the order of integration (and the substitution $y = x + \lambda\rho u' \in Q_{2r}$) we obtain that (4.2.13) is bounded by

$$\int_0^1 \int_{\mathbb{S}^{n-1}} \int_0^r \frac{1}{|Q_r|} \int_{Q_{2r}} \left|\left(M\left(\nabla u, \tilde{Q}_\rho(y,t)\right) - M\left(\nabla u, \tilde{Q}_\rho(y - \rho e_k, t)\right)\right)\right|^2 \mathrm{d}y\, \mathrm{d}t \frac{\mathrm{d}\rho}{\rho}\, \mathrm{d}u\, \mathrm{d}\lambda.$$

By lemma 4.2.6 the two interior integrals are bounded by $C\|\nabla\phi\|_*^2$. Therefore (4.2.13) is controlled by $C\|\nabla\phi\|_*^2$. $\square$

The opposite implications are likely to be false due the highly singular nature of Riesz potentials, c.f. (4.2.7) and (4.2.8). Even the spatial Riesz potentials $\widehat{R_j^{-1}}$ are not just singular at a point but along a co-dimension 1 hypersurface.

Corollary 4.2.10. *If* $\|\nabla\phi\|_* \lesssim \eta$ *and* $B_{(5.b)} \lesssim \eta^2$ *then* $\|\mathbb{D}\phi\|_* \lesssim \eta$.

Here we have replaced conditions (5.a) or (6.a) by the slightly stronger but easier to verify condition $\|\nabla\phi\|_* \lesssim \eta$. We believe that, without too much extra work, one could formulate our main theorem and associated lemmas with a local version of condition (5.a) in place of $\|\nabla\phi\|_*$.

Remark 4.2.11. In [Riv03, Lemma 2.1] it is stated that another condition is equivalent to those given in theorems 4.2.5 and 4.2.7; however this claim is not correct and only one of the stated implications holds.

A result of Strichartz [Str80, Theorem 3.3] states that in the one dimensional setting $D^t_{1/2}\phi(t) \in \mathrm{BMO}(\mathbb{R})$ is equivalent to the one dimensional version of conditions (5.b) and (6.b)

$$\sup_{I' \subset \mathbb{R}} \left(\frac{1}{|I'|} \int_{I'} \int_{I'} \frac{|\phi(t) - \phi(s)|^2}{|t-s|^2} \,\mathrm{d}t\,\mathrm{d}s \right)^{1/2} \leq B \tag{4.2.26}$$

with $B \sim \|D^t_{1/2}\phi(\cdot)\|_{\mathrm{BMO}(\mathbb{R})}$.

In [Riv03, Lemma 2.1] it is claimed that given $\phi : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}$ and $\phi \in \mathrm{Lip}(1, 1/2)$ the pointwise n-dimensional analogue of (4.2.26)

$$\sup_{x \in \mathbb{R}^{n-1}} \sup_{I' \subset \mathbb{R}} \left(\frac{1}{|I'|} \int_{I'} \int_{I'} \frac{|\phi(x,t) - \phi(x,s)|^2}{|t-s|^2} \,\mathrm{d}t\,\mathrm{d}s \right)^{1/2} \leq B \tag{4.2.27}$$

is equivalent to $\mathbb{D}\phi \in \mathrm{BMO}(\mathbb{R}^n)$ with $B \sim \|\mathbb{D}\phi\|_{\mathrm{BMO}(\mathbb{R}^n)}$. This is incorrect. By [Str80] equation (4.2.27) is equivalent to $D^t_{1/2}\phi(x, \cdot) \in \mathrm{BMO}(\mathbb{R})$ pointwise for a.e. x. After some tedious and technical calculations we were able to show $\sup_x D^t_{1/2}\phi(x, \cdot) \in \mathrm{BMO}(\mathbb{R})$ implies $D^t_{1/2}\phi \in \mathrm{BMO}(\mathbb{R}^n)$ and hence $\mathbb{D}\phi \in \mathrm{BMO}(\mathbb{R}^n)$ via condition (4) of theorem 4.2.7. However, the converse is not true even if we assume more structure for the function $\mathbb{D}\phi$. This is due to the fact that there is "no reasonable Fubini theorem relating $\mathrm{BMO}(\mathbb{R}^n)$ to $\mathrm{BMO}(\mathbb{R})$" [Str80, p. 558].

Fortunately the lack of a converse implication does not cast doubt over the subsequent results of [Riv03] since the author only uses the claimed equivalence in the correct direction — that (4.2.27) implies $\mathbb{D}\phi \in \mathrm{BMO}(\mathbb{R}^n)$.

Remark 4.2.12. Let us recall the definition of $D^t_{1/2}\phi$ for $\phi : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}$ from (4.2.4):

$$D^t_{1/2}\phi(x,t) = c_n \int_{\mathbb{R}} \frac{\phi(x,s) - \phi(x,t)}{|s-t|^{3/2}} \,\mathrm{d}s.$$

In [DH18; DPP17] they state that the following is a local version of this. Given a bounded interval $I \subset \mathbb{R}$ and $\tilde{\phi}$ defined on $\{x\} \times I$ then they define $D^t_{1/2}\tilde{\phi}$ as

$$D^t_{1/2}\tilde{\phi}(x,t) = c_n \int_I \frac{\tilde{\phi}(x,s) - \tilde{\phi}(x,t)}{|s-t|^{3/2}} \,\mathrm{d}s, \quad \text{for all } t \in I. \tag{4.2.28}$$

It is not obvious that this definition is the appropriate localisation or that one can extend a local function $\tilde{\phi}$ to a global function ϕ whilst still preserving the BMO norm of $D^t_{1/2}\tilde{\phi}$. This extension is thought not to hold. This is due to the cancellation that occurs within the non-truncated operator and the BMO norm being strongly influenced by cancellation.

4.2.3 Localisation

After the comprehensive review of the Lewis-Murray condition for a graph domain Ω we continue in our aim to construct a time-varying domain which is locally described by local graphs ϕ_j. To this end we use $\nabla\phi$, which is a local operator, and localise condition (5.b) of theorem 4.2.7. We discuss uniform closeness to VMO, introduce some helpful notation and prove an extension result. We then define admissible and VMO-type domains, and compare their definitions to Lewis-Murray cylinders.

For a vector $x \in \mathbb{R}^{n-1}$ we denote the norm $|x|_\infty = \sup_i |x_i|$.

Consider $\phi : Q_{8d} \to \mathbb{R}^{n-1} \times \mathbb{R}$. The localised version of (4.2.12) from theorem 4.2.7 is simply

$$\sup_{\substack{Q_r = J_r \times I_r \\ Q_r \subset Q_{8d}}} \frac{1}{|Q_r|} \int_{Q_r} \int_{I_r} \frac{|\phi(x,t) - \phi(x,s)|^2}{|t-s|^2} \,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}x < \infty. \tag{4.2.29}$$

Furthermore, we wish to construct time-varying domains that are close to VMO. We explain VMO-type domains later in definition 4.2.20 but think of these as domains where the derivative in space and half derivative in time (or Lewis-Murray condition) are close to VMO. An intuitive way to achieve this, for the half derivative in time, is to ask that there exists a scale $r_1 > 0$ and constant $\eta > 0$ such that

$$\sup_{\substack{Q_s = J_s \times I_s \\ Q_s \subset Q_{8d},\, s \le r_1}} \frac{1}{|Q_s|} \int_{Q_s} \int_{I_s} \frac{|\phi(x,t) - \phi(x,\tau)|^2}{|t-\tau|^2} \, \mathrm{d}\tau \, \mathrm{d}t \, \mathrm{d}x \le \eta^2. \tag{4.2.30}$$

We do not prove $\eta \to 0$ and $r_1 \to 0$ in (4.2.30) and $\nabla\phi \in \mathrm{VMO}$ is equivalent to $\mathbb{D}\phi \in \mathrm{VMO}$ or $D^t_{1/2}\phi \in \mathrm{VMO}$, however we would expect something like to this to hold.

We write $\|f\|_{*,d}$ to be the BMO norm of f where the supremum in the BMO norm, c.f. (2.5.3), is taken over all cubes Q_r with $r \le d$. For a function $f : J \times I \to \mathbb{R}$, where $J \subset \mathbb{R}^{n-1}$ and $I \subset \mathbb{R}$ are closed bounded cubes, we consider the norm $\|f\|_{*,J\times I}$ defined as above where the supremum is taken over all parabolic cubes Q_r contained in $J \times I$. The norm $\|f\|_{*,J\times I,d}$ is where the supremum is taken over all parabolic cubes Q_r with $r \le d$ contained in $J \times I$. If the context is clear we suppress the $J \times I$ and just write $\|f\|_*$ or $\|f\|_{*,d}$.

Recall that $\mathrm{VMO}(\mathbb{R}^n)$ is defined as the closure of all compactly supported continuous functions in the BMO norm or equivalently BMO functions f such that $\|f\|_{*,d} \to 0$ as $d \to 0$. Alternatively, if we define

$$d(f, \mathrm{VMO}) := \inf_{h \in C_c} \|f - h\|_*$$

then $f \in \mathrm{VMO}$ if and only if $d(f, \mathrm{VMO}) = 0$; for $f \in \mathrm{BMO}$ this measures the distance of f to VMO. In our case, the boundary of the parabolic domains we consider can be locally described as a graph of a continuous function. However, as our domain is unbounded in time we may potentially require an infinite family of local graphs $\{\phi_j\}$. Therefore we need to measure the distance to VMO uniformly across this infinite family.

Let $\delta : \mathbb{R}_+ \to \mathbb{R}_+$, $\delta(0) = 0$ and δ be continuous at 0 then we define C_δ to be the set of continuous functions with the same modulus of continuity δ. That is

$$C_\delta = \{g \in C : |g(x) - g(y)| \le \delta(|x-y|) \text{ for all } x, y\}. \tag{4.2.31}$$

Note that every family of equicontinuous functions can be represented as C_δ for some function δ and $C = \cup_\delta C_\delta$. Here the compact support is implicit. For $f : Q_{8d} \to \mathbb{R}$ we define $d(f, C_\delta)$ as

$$d(f, C_\delta) = \inf_{h \in C_\delta} \|f - h\|_{*,Q_{8d}}.$$

We are now ready to state and prove the following result on the extendability of $\phi : Q_{8d} \to \mathbb{R}$ to a global function. This is because later on we use lemma 4.2.26 that needs our local graphs ϕ to be global functions.

Theorem 4.2.13. *Let $\phi : Q_{8d} \subset \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}$ be* $\mathrm{Lip}(1,1/2)$ *with Lipschitz constant ℓ. If there exists a scale r_1, a constant $\eta > 0$ and a modulus of continuity δ such that*

$$\sup_{\substack{Q_s = J_s \times I_s \\ Q_s \subset Q_{8d},\, s \le r_1}} \frac{1}{|Q_s|} \int_{Q_s} \int_{I_s} \frac{|\phi(x,t) - \phi(x,\tau)|^2}{|t-\tau|^2} \, \mathrm{d}\tau \, \mathrm{d}t \, \mathrm{d}x \le \eta^2 \tag{4.2.32}$$

and

$$d(\nabla\phi, C_\delta) \le \eta \tag{4.2.33}$$

then there exists a scale $d' \le d$ (that only depends on d, δ, η, and r_1 and not ϕ) such that for all $Q_r \subset Q_{4d}$ with $r \le d'$ there exists a global $\mathrm{Lip}(1,1/2)$ *function $\Phi : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}$ with the following properties for all $0 < \varepsilon < 1$:*

(i) $\Phi|_{Q_r} = \phi|_{Q_r}$.

(ii) The $\mathrm{Lip}(1,1/2)$ *constant of Φ is ℓ.*

(iii) $\|\nabla\Phi\|_* \lesssim_\varepsilon \eta^{1-\varepsilon} + \eta\ell$.

(iv) $\displaystyle \sup_{Q_s = J_s \times I_s} \frac{1}{|Q_s|} \int_{Q_s} \int_{I_s} \frac{|\Phi(x,t) - \Phi(x,\tau)|^2}{|t-\tau|^2} \, \mathrm{d}\tau \, \mathrm{d}t \, \mathrm{d}x \lesssim \eta^2.$

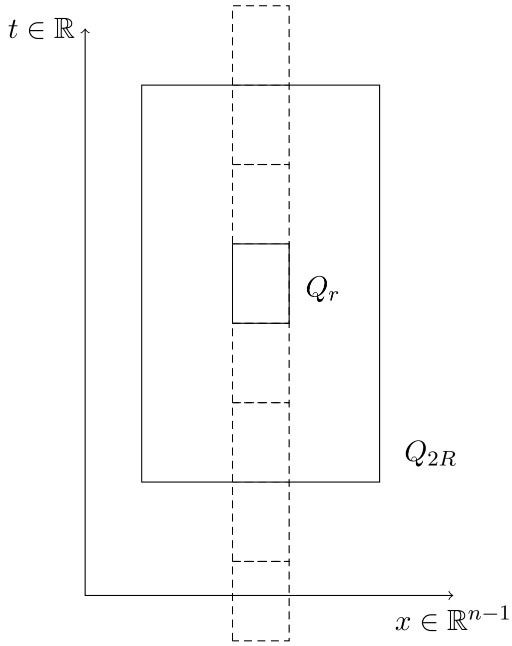

Figure 4.2: The reflection and tiling of the cube $Q_r \subset Q_{2R}$ defined in (4.2.34).

Therefore by corollary 4.2.10, $\|\mathbb{D}\Phi\|_* \lesssim_\varepsilon \eta^{1-\varepsilon} + \eta\ell$.

Proof. Without loss of generality we only consider the case $\eta < 1$. When $\eta \geq 1$ the existence of a extension with $\|\mathbb{D}\Phi\|_* \lesssim \eta + \ell$ requires a much simpler argument.

By (4.2.33) there exists $f \in C_\delta$ such that $\|\nabla\phi - f\|_{*,Q_{8d}} \leq 2\eta$ and a scale $0 < r_0 = r_0(\delta) \leq d$ such that

$$\|f\|_{*,Q_{8d},r_0} \leq 2\eta.$$

Let $d' = \eta \min(r_0, r_1)/2$ and consider some $r \leq d'$ and $Q_r \subset Q_{4d}$. We find a natural number k such that $R = 2^k r$ and $R\eta/2 < r \leq R\eta$. By our choice of d' the cube Q_{2R}, which is an enlargement of Q_r by a factor 2^{k+1}, is still contained in the original cube Q_{8d}.

It follows that

$$\|\nabla\phi\|_{*,Q_{2R}} \lesssim \eta \quad \text{and}$$
$$\sup_{\substack{Q_s = J_s \times I_s \\ Q_s \subset Q_{2R}}} \frac{1}{|Q_s|} \int_{Q_s} \int_{I_s} \frac{|\phi(x,t) - \phi(x,\tau)|^2}{|t-\tau|^2} \, \mathrm{d}\tau \, \mathrm{d}t \, \mathrm{d}x \leq \eta^2.$$

Without loss of generality we may assume that the cube Q_{2R} is centred at the origin $(0,0)$ and that $\phi(0,0) = 0$, since the BMO norm is invariant under translation and ignores constants. We first define $\tilde{\phi}$ as an extension in time via reflection and tiling of the cube Q_r:

$$\tilde{\phi}(x,t) = \begin{cases} \phi(x,t) & t \in [-r^2, r^2] + 4kr^2, \\ \phi(x, 2r^2 - t) & t \in [r^2, 3r^2] + 4kr^2, \; k \in \mathbb{Z}. \end{cases} \tag{4.2.34}$$

See figure 4.2 on p. 58 for an illustration of this. Clearly $\tilde{\phi}$ coincides with ϕ on Q_r.

It follows that $\tilde{\phi}$ is a function $\tilde{\phi} : \{|x|_\infty < 2R\} \times \mathbb{R} \to \mathbb{R}$ and $(\nabla\tilde{\phi})_{Q_r} = (\nabla\phi)_{Q_r}$. Consider a cut off function ρ such that

$$\rho(x) = \begin{cases} 1 & \text{if } |x|_\infty < r, \\ 0 & \text{if } |x|_\infty > 2R, \end{cases}$$

and $|\nabla\rho| \lesssim 1/R \lesssim \eta/r$. Finally define

$$\Phi = \tilde{\phi}\rho + (1-\rho)(x \cdot (\nabla\tilde{\phi})_{Q_r}). \tag{4.2.35}$$

Clearly Φ is well defined on $\mathbb{R}^{n-1} \times \mathbb{R}$ as $\rho = 0$ outside the support of $\tilde{\phi}$. We claim that Φ satisfies properties (i) to (iv) of theorem 4.2.13. We establish this in a sequence of lemmas below. Observe also that from our definition of Φ we have

$$\nabla\Phi = \left(\nabla\tilde{\phi} - (\nabla\tilde{\phi})_{Q_r}\right)\rho + \nabla\rho(\tilde{\phi} - x\cdot(\nabla\tilde{\phi})_{Q_r}) + (\nabla\tilde{\phi})_{Q_r}. \tag{4.2.36}$$

We start with couple of lemmas that allow us to reduce our claim to the dyadic case; this is to make the geometry easier to handle.

Lemma 4.2.14 ([Jon80, Lemma 2.3], c.f. [Str80, Theorem 2.8])**.** *Let f be defined on $\mathbb{R}^n$ and*

$$\sup_Q \frac{1}{|Q|}\int_Q |f - f_Q| \le c(\eta), \tag{4.2.37}$$

where the supremum is taken over all dyadic cubes $Q \subset \mathbb{R}^n$. Further, assume that

$$\sup_{Q_1,Q_2} |f_{Q_1} - f_{Q_2}| \le c(\eta), \tag{4.2.38}$$

where the supremum is taken over all dyadic cubes Q_1, Q_2 of equal edge length with a touching edge then

$$\|f\|_* \lesssim c(\eta).$$

Below $l(Q_s) = s$ denotes the radius of a parabolic cube.

Lemma 4.2.15 ([Jon80, Lemma 2.1 and pp. 44-45])**.** *Let $f \in \mathrm{BMO}(Q)$ and $Q_0 \subset Q_1 \subset Q$ then*

$$|f_{Q_0} - f_{Q_1}| \lesssim \log\left(2 + \frac{l(Q_1)}{l(Q_0)}\right)\|f\|_{*,Q}. \tag{4.2.39}$$

Furthermore, the same proof in [Jon80] gives the following slightly stronger result

$$\frac{1}{|Q_0|}\int_{Q_0}|f - f_{Q_1}| \lesssim \log\left(2 + \frac{l(Q_1)}{l(Q_0)}\right)\|f\|_{*,Q}. \tag{4.2.40}$$

If $Q_0, Q_1 \subset Q$ and $l(Q_0) \le l(Q_1)$ but they are not necessarily nested then

$$|f_{Q_0} - f_{Q_1}| \lesssim \left(\log\left(2 + \frac{l(Q_1)}{l(Q_0)}\right) + \log\left[2 + \frac{\mathrm{dist}(Q_0,Q_1)}{l(Q_1)}\right]\right)\|f\|_{*,Q}. \tag{4.2.41}$$

If the cubes Q_0, Q_1 and Q are dyadic then we may replace BMO *by dyadic* BMO*.*

There is a typo at the top of [Jon80, p. 45]. It should read $l(Q_k) \le l(Q_j)$ (it currently reads the converse).

Claim 4.2.16. *Let $\tilde{\phi}$ be defined as in* (4.2.34)*, $\|\nabla\phi\|_{*,Q_{2R}} \lesssim \eta$, and let Q be dyadic with $r \le l(Q) \le 2R$ then*

$$\frac{1}{|Q|}\int_Q |\nabla\tilde{\phi} - \nabla\tilde{\phi}_{Q_r}| \lesssim_\varepsilon \eta^{1-\varepsilon}. \tag{4.2.42}$$

Proof of claim. Let $N \in \mathbb{N}$ be such that $l(Q) = 2^N l(Q_r)$. Let $\{Q^i\}$ be the $2^{N(n-1)}$ dyadic cubes that are translations of Q_r and partition $Q \cap \{|t| \le r^2\}$. Then by lemma 4.2.15

$$\begin{aligned}\frac{1}{|Q|}\int_Q |\nabla\tilde{\phi} - \nabla\tilde{\phi}_{Q_r}| &= \sum_i \frac{2^{2N}|Q^i|}{|Q|}\frac{1}{|Q^i|}\int_{Q^i}|\nabla\tilde{\phi} - \nabla\tilde{\phi}_{Q_r}| \\ &\le \sum_i \frac{2^{2N}|Q^i|}{|Q|}\left(\frac{1}{|Q^i|}\int_{Q^i}|\nabla\phi - \nabla\phi_{Q^i}| + |\nabla\phi_{Q^i} - \nabla\phi_{Q_r}|\right) \\ &\lesssim (\eta + \eta\log(2 + R/r)) \lesssim \eta + \eta\log(1 + 1/\eta) \lesssim_\varepsilon \eta^{1-\varepsilon}. \qquad \square\end{aligned}$$

Lemma 4.2.17 ([Ste76]). *Let $g, h \in L^1_{\mathrm{loc}}$ then*

$$\frac{1}{|Q|}\int_Q |gh-(gh)_Q| \leq \frac{2}{|Q|}\int_Q |g(h-h_Q)| + \frac{|h_Q|}{|Q|}\int_Q |g-g_Q|. \tag{4.2.43}$$

Proof. This small reduction is from [Ste76, p. 582]. First observe

$$gh-(gh)_Q = g(h-h_Q)+h_Q(g-g_Q)+g_Qh_Q-(gh)_Q$$

and

$$|g_Qh_Q-(gh)_Q| = \left|\frac{1}{|Q|}\int_Q gh_Q - \frac{1}{|Q|}\int_Q gh\right| \leq \frac{1}{|Q|}\int_Q |g(h-h_Q)|$$

hence

$$\left|\frac{1}{|Q|}\int_Q |gh-(gh)_Q| - \frac{|h_Q|}{|Q|}\int_Q |g-g_Q|\right| \leq 2\frac{1}{|Q|}\int_Q |g(h-h_Q)|. \tag{4.2.44}$$

□

We can now prove property (iii) of theorem 4.2.13.

Lemma 4.2.18. *Let $\Phi : \mathbb{R}^n \to \mathbb{R}$ be defined as in* (4.2.35) *with $\|\nabla\phi\|_{*,Q_{2R}} \lesssim \eta$ then $\nabla\Phi \in \mathrm{BMO}(\mathbb{R}^n)$ and for all $0<\varepsilon<1$*

$$\|\nabla\Phi\|_* \lesssim_\varepsilon \eta^{1-\varepsilon} + \eta\ell. \tag{4.2.45}$$

Proof. Recall $\nabla\Phi = \left(\nabla\tilde{\phi} - (\nabla\tilde{\phi})_{Q_r}\right)\rho + \nabla\rho(\tilde{\phi} - x\cdot(\nabla\tilde{\phi})_{Q_r}) + (\nabla\tilde{\phi})_{Q_r}$; we can ignore the constant term as the BMO norm does not see it. Let $\psi = \nabla\tilde{\phi} - (\nabla\tilde{\phi})_{Q_r}$ and $\theta = \tilde{\phi} - x\cdot(\nabla\tilde{\phi})_{Q_r}$. We want to bound $\|\rho\psi\|_*$ and $\|\nabla\rho\theta\|_*$. We show this by lemma 4.2.14. First we tackle the term $\|\rho\psi\|_*$.

Step 1: (4.2.38) holds: $\sup_{Q_1,Q_2} |(\rho\psi)_{Q_1} - (\rho\psi)_{Q_2}| \leq c(\eta)$ for Q_1, Q_2 dyadic cubes of equal side length and with a touching edge.

Since $\tilde{\phi}$ is the extension in the time direction by reflection and tiling (c.f. (4.2.34)), and Q_1, Q_2 and Q_r are all dyadic cubes we may assume that if $l(Q_1) \leq r$ then $Q_1, Q_2 \subset \{|t| < r^2\}$, and if $l(Q_1) > r$ then $\{|t| < r^2\} \subset Q_1$.

If $Q_1, Q_2 \subset Q_{2R}$ then by lemma 4.2.15 $|(\rho\psi)_{Q_1} - (\rho\psi)_{Q_2}| \lesssim \|\rho\psi\|_{*,\,\mathrm{dyadic},\,Q_{2R}}$. Therefore this case reduces down to controlling $\|\rho\psi\|_{*,\mathrm{dyadic},Q_{2R}}$ which is shown in step 2 below.

Now look at the other cases: $Q_1 \subset Q_{2R}$ and $Q_2 \cap Q_{2R} = \emptyset$, or $Q_{2R} \subset Q_1$ and $Q_2 \cap Q_{2R} = \emptyset$. In both cases we wish to control $|(\rho\psi)_{Q_1}|$.

Step 1.a: Case $Q_1 \subset Q_{2R}$, $Q_2 \cap Q_{2R} = \emptyset$ and $l(Q_1) \lesssim \frac{R\eta}{\ell}$.

Q_1 is small here and touches the boundary of Q_{2R}. This means that $\|\rho\|_{L^\infty(Q_1)} \lesssim \frac{l(Q_1)}{R}$ since ρ is 0 outside Q_{2R}. Therefore we just apply the trivial bound

$$|(\rho\psi)_{Q_1}| \leq \|\rho\|_{L^\infty(Q_1)}\|\psi\|_{L^\infty(Q_1)} \lesssim \frac{l(Q_1)}{R}\ell \lesssim \eta.$$

Step 1.b: Case $Q_1 \subset Q_{2R}$, $Q_2 \cap Q_{2R} = \emptyset$ and $\frac{R\eta}{\ell} \lesssim l(Q_1) \leq 2R$.

Since $Q_1 \subset Q_{2R}$ we have $\frac{R\eta}{\ell} \lesssim l(Q_1) \leq 2R$. Q_1 is dyadic so there exists $N \in \mathbb{Z}$ such that $l(Q_1) = 2^N l(Q_r)$.

Step 1.b.i: $N \leq 0$:
This means that $l(Q_1) \leq l(Q_r)$ and so by the reflection and tiling in time, (4.2.34), we may assume $Q_1 \subset \{|t| \leq r^2\}$. By lemma 4.2.15

$$\begin{aligned}|(\rho\psi)_{Q_1}| \leq |\psi|_{Q_1} = \frac{1}{|Q_1|}\int_{Q_1} |\nabla\phi - \nabla\phi_{Q_r}| &\leq \frac{1}{|Q_1|}\int_{Q_1} |\nabla\phi - \nabla\phi_{Q_1}| + |\nabla\phi_{Q_1} - \nabla\phi_{Q_r}| \\ &\lesssim \eta + \eta\log(1+\ell) + \eta\log(1+1/\eta) \lesssim_\varepsilon \eta^{1-\varepsilon} + \eta\log(1+\ell).\end{aligned}$$

Step 1.b.ii: $N > 0$:
By claim 4.2.16 we obtain

$$|(\rho\psi)_{Q_1}| \leq |\psi|_{Q_1} = \frac{1}{|Q_1|}\int_{Q_1} |\nabla\tilde{\phi} - \nabla\tilde{\phi}_{Q_r}| \lesssim_\varepsilon \eta^{1-\varepsilon}.$$

Step 1.c: Case $Q_{2R} \subset Q_1$, $Q_2 \cap Q_{2R} = \emptyset$ so $l(Q_1) \geq 2R$.

Let N satisfy $l(Q_1) = 2^N l(Q_{2R})$, the number of dyadic generations separating Q_1 and Q_{2R}. Then Q_1 overlaps Q_{2R} (and its dyadic translates in time) exactly 2^{2N} times. Therefore by claim 4.2.16

$$|(\rho\psi)_{Q_1}| \leq |\psi|_{Q_1} \leq \frac{2^{2N}}{|Q_1|}\int_{Q_{2R}} |\nabla\tilde{\phi} - \nabla\tilde{\phi}_{Q_r}| \leq \frac{2^{2N}}{2^{N(n+1)}}\frac{1}{|Q_{2R}|}\int_{Q_{2R}} |\nabla\tilde{\phi} - \nabla\tilde{\phi}_{Q_r}| \lesssim_\varepsilon \eta^{1-\varepsilon}.$$

Hence, modulo the unproven statement $\|\rho\psi\|_{*,\,\text{dyadic},\,Q_{2R}} \lesssim_\varepsilon \eta^{1-\varepsilon}$ we have shown

$$|(\rho\psi)_{Q_1} - (\rho\psi)_{Q_2}| \lesssim_\varepsilon \eta^{1-\varepsilon} + \eta\log(1+\ell).$$

Step 2: (4.2.37) holds, that is: $\|\rho\psi\|_{*,\,\text{dyadic}} \lesssim c(\eta)$.

By applying lemma 4.2.17 we need to control the two terms below:

$$\sup_{Q\text{ dyadic}} \|\rho\|_{L^\infty(Q)}\frac{1}{|Q|}\int_Q |\psi - \psi_Q|,$$
$$\sup_{Q\text{ dyadic}} \frac{|\psi_Q|}{|Q|}\int_Q |\rho - \rho_Q|.$$

Step 2.a: Estimating $\sup_{Q\text{ dyadic}} \|\rho\|_{L^\infty(Q)}\frac{1}{|Q|}\int_Q |\psi - \psi_Q|$.

In all the following sub-cases we bound $\|\rho\|_{L^\infty(Q)} \leq 1$.

Step 2.a.i: Case $l(Q) \leq r$.
As before, by the reflection and tiling in time, we may assume $Q \subset \{|t| \leq r^2\}$ and so $\nabla\tilde{\phi} = \nabla\phi$ on Q. Hence

$$\frac{1}{|Q|}\int_Q |\psi - \psi_Q| = \frac{1}{|Q|}\int_Q |\nabla\tilde{\phi} - (\nabla\tilde{\phi})_Q| = \frac{1}{|Q|}\int_Q |\nabla\phi - (\nabla\phi)_Q| \lesssim \eta.$$

Step 2.a.ii: Case $r < l(Q) \leq 2R$.
Applying claim 4.2.16 gives

$$\frac{1}{|Q|}\int_Q |\psi - \psi_Q| \leq |\psi|_Q \lesssim_\varepsilon \eta^{1-\varepsilon}.$$

Step 2.a.iii: Case $2R < l(Q)$.
From step 1.c it follows that

$$\frac{1}{|Q|}\int_Q |\psi - \psi_Q| \leq |\psi|_Q \lesssim_\varepsilon \eta^{1-\varepsilon}.$$

Step 2.b: Estimating $\sup_{Q\text{ dyadic}} \frac{|\psi_Q|}{|Q|}\int_Q |\rho - \rho_Q|$.

We have the following three cases to consider.

Step 2.b.i: Case $Q \subset Q_{2R}$, $l(Q) \leq r$ and $Q \subset \{|t| \leq r^2\}$.
Because the cube Q might not be touching the boundary we can not just follow step 2.a and bound $\frac{1}{|Q|}\int_Q |\rho - \rho_Q|$ by $\|\rho\|_{L^\infty(Q)}$, which here is likely be 1. However, we can use the mean value theorem and get a better bound. By the intermediate value theorem there exists

$(z,\tau) \in Q$ such that $\rho(z) = \rho_Q$. Using that ρ is independent of time and $|\nabla\rho| \lesssim 1/R$ we have

$$|\rho(x) - \rho_Q| = |\rho(x) - \rho(z)| \leq |\nabla\rho| l(Q) \lesssim \frac{l(Q)}{R} \leq \frac{l(Q)}{r}.$$

Then applying lemma 4.2.15 gives

$$\begin{aligned}\frac{|\psi_Q|}{|Q|}\int_Q |\rho - \rho_Q| &\lesssim \frac{l(Q)}{r}\left|\frac{1}{|Q|}\int_Q \nabla\tilde{\phi} - \nabla\tilde{\phi}_{Q_r}\right| \leq \frac{l(Q)}{r}\frac{1}{|Q|}\int_Q |\nabla\phi - \nabla\phi_{Q_r}| \\ &\lesssim \frac{l(Q)}{r}\log\left(2 + \frac{r}{l(Q)}\right)\eta \lesssim \eta.\end{aligned}$$

Step 2.b.ii: Case $Q \subset Q_{2R}$ and $r < l(Q) \leq 2R$.
This case is a straightforward application of claim 4.2.16

$$\frac{|\psi_Q|}{|Q|}\int_Q |\rho - \rho_Q| \leq |\psi_Q| \lesssim_\varepsilon \eta^{1-\varepsilon}.$$

Step 2.b.iii: Case $Q_{2R} \subset Q$ so $l(Q) > 2R$.
This follows similarly to step 1.c; let N be the number of dyadic generations defined there then

$$\frac{|\psi_Q|}{|Q|}\int_Q |\rho - \rho_Q| \leq \frac{1}{|Q|}\left|\int_Q \nabla\phi - \nabla\phi_{Q_{2R}}\right| \leq \frac{2^{2N}}{2^{N(n+1)}}\|\nabla\phi\|_{*,Q_{2R}} \leq \eta.$$

Therefore by lemma 4.2.14, $\|\rho\psi\|_* \lesssim_\varepsilon \eta^{1-\varepsilon} + \eta\log(1+\ell)$.

It remains to tackle the harder piece $\nabla\rho\theta = \nabla\rho(\tilde{\phi} - x\cdot\nabla\tilde{\phi}_{Q_r})$.

Step 3: (4.2.38) holds: $\sup_{Q_1,Q_2} |(\nabla\rho\theta)_{Q_1} - (\nabla\rho\theta)_{Q_2}| \leq c(\eta)$ where Q_1, Q_2 are dyadic with a touching edge and $l(Q_1) = l(Q_2)$.

Recall $\mathrm{supp}(\nabla\rho) = \{r \leq |x|_\infty \leq 2R\}$. There are two different cases to consider:

(1) $Q_1 \cap \mathrm{supp}(\nabla\rho) \neq \emptyset$ and $Q_2 \cap \mathrm{supp}(\nabla\rho) \neq \emptyset$.

(2) $Q_1 \cap \mathrm{supp}(\nabla\rho) \neq \emptyset$ and $Q_2 \cap \mathrm{supp}(\nabla\rho) = \emptyset$.

Again case (1) is controlled by $\|\nabla\rho\theta\|_{*,\,\mathrm{dyadic},\,Q_{2R}}$ by lemma 4.2.15. So we only have to deal with case (2) and bound $\sup\limits_{Q_1 \text{ dyadic}} |(\nabla\rho\theta)_{Q_1}|$.

Step 3.a: Case $Q_1 \subset Q_{2R}$ and $l(Q_1) \lesssim \frac{R\eta}{\ell}$.
In this case Q_1 touches the boundary of the support of $\nabla\rho$ so we have the estimate $\|\nabla\rho\|_{L^\infty(Q_1)} \lesssim \frac{l(Q_1)}{R^2}$ since $|\nabla^2\rho| \lesssim 1/R^2$. Also $\phi(0,0) = 0$ and $\phi \in \mathrm{Lip}(1,1/2)$ hence $\|\tilde{\phi}(x,t)\|_{L^\infty(Q_1)} \leq \|\phi(x,t)\|_{L^\infty(Q_{2R})} \lesssim \ell R$; finally $\|x\cdot\nabla\tilde{\phi}_{Q_r}\|_{L^\infty(Q_{2R})} \lesssim \ell R$. Therefore

$$\begin{aligned}|(\nabla\rho\theta)_{Q_1}| \leq \|\nabla\rho\|_{L^\infty(Q_1)}|\theta|_{Q_1} &\lesssim \frac{l(Q_1)}{R^2}\frac{1}{|Q_1|}\int_{Q_1} |\tilde{\phi}(x,t) - x\cdot\nabla\tilde{\phi}_{Q_r}|\,\mathrm{d}x\,\mathrm{d}t \\ &\lesssim \frac{l(Q_1)}{R^2}\ell R \lesssim \eta.\end{aligned}$$

Step 3.b: Case $Q_1 \subset Q_{2R}$ and $\frac{R\eta}{\ell} \lesssim l(Q_1) \leq 2R$.

By the fundamental theorem of calculus we may write

$$\tilde{\phi}(x,t) - \tilde{\phi}\left(r\frac{x}{|x|}, t\right) = x\cdot\int_{r/|x|}^1 \nabla\tilde{\phi}(\lambda x, t)\,\mathrm{d}\lambda.$$

Therefore since $r \sim R\eta$

$$
\begin{aligned}
|(\nabla\rho\theta)_{Q_1}| &\le |\nabla\rho||\theta|_{Q_1} \\
&= |\nabla\rho| \left| \tilde{\phi}\left(r\frac{x}{|x|}, t\right) + x \cdot \int_{r/|x|}^{1} \left(\nabla\tilde{\phi}(\lambda x, t) - \nabla\tilde{\phi}_{Q_r}\right) \mathrm{d}\lambda + x \cdot \frac{r}{|x|}\nabla\tilde{\phi}_{Q_r} \right|_{Q_1} \\
&\lesssim \frac{1}{R}\left\| \tilde{\phi}\left(r\frac{x}{|x|}, t\right)\right\|_{L^\infty(Q_1)} + \frac{R}{R}\frac{1}{|Q_1|}\int_{Q_1}\left(\int_{r/|x|}^{1} |\nabla\tilde{\phi}(\lambda x, t) - \nabla\tilde{\phi}_{Q_r}|\,\mathrm{d}\lambda\right)\mathrm{d}x\,\mathrm{d}t \\
&\quad + \frac{R^2\eta\ell}{R^2}.
\end{aligned}
$$

Since $\tilde{\phi}$ defined by (4.2.34) is tiled and reflected in time on cubes of scale r, and $(rx/|x|, 0) \in Q_r$ we control the first term above by

$$
\frac{1}{R}\left\| \tilde{\phi}\left(r\frac{x}{|x|}, t\right) - 0\right\|_{L^\infty(Q_1)} \le \frac{1}{R}\|\phi - \phi(0,0)\|_{L^\infty(Q_r)} \lesssim \frac{\ell r}{R} \lesssim \ell\eta.
$$

Recall that $r \sim \eta R$, $\frac{R\eta}{\ell} \lesssim l(Q_1) \le 2R$ and $r \le |x|_\infty \le 2R$ so $\eta/2 \le \lambda \le 1$. We apply Fubini to the second term

$$
\begin{aligned}
&\frac{1}{|Q_1|}\int_{Q_1}\left(\int_{r/|x|}^{1} |\nabla\tilde{\phi}(\lambda x, t) - \nabla\tilde{\phi}_{Q_r}|\,\mathrm{d}\lambda\right)\mathrm{d}x\,\mathrm{d}t \\
&\qquad \le \frac{1}{|Q_1|}\int_{\eta/2}^{1}\int_{Q_1} |\nabla\tilde{\phi}(\lambda x, t) - \nabla\tilde{\phi}_{Q_r}|\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\lambda.
\end{aligned}
$$

Let $\tilde{Q}_1$ be the set formed by Q_1 under the transformation $(x, t) \mapsto (\lambda x, t)$. We may further cover $\tilde{Q}_1$ by $\sim \lambda^{-2}$ translations of λQ_1 with $|\lambda Q_1|/|\tilde{Q}_1| \lesssim \lambda^2$. Therefore a similar argument to claim 4.2.16 (using lemma 4.2.15) gives

$$
\begin{aligned}
&\frac{1}{|Q_1|}\int_{Q_1} |\nabla\tilde{\phi}(\lambda x, t) - \nabla\tilde{\phi}_{Q_r}|\,\mathrm{d}x\,\mathrm{d}t = \frac{1}{|\tilde{Q}_1|}\int_{\tilde{Q}_1} |\nabla\tilde{\phi} - \nabla\tilde{\phi}_{Q_r}| \\
&\lesssim \lambda^{-2}\frac{\lambda^2}{|sQ_1|}\int_{sQ_1} |\nabla\tilde{\phi} - \nabla\tilde{\phi}_{Q_r}| \lesssim \eta\log\left(2 + \frac{r}{sl(Q_1)}\right) \lesssim \eta\log\left(1 + \frac{\ell}{\eta^2}\right) \\
&\lesssim_\varepsilon \eta^{1-\varepsilon} + \eta\log(1+\ell)
\end{aligned}
$$

and hence after harmlessly integrating in λ we can control the second term by:

$$
\int_{\eta/2}^{1} \eta\log\left(1 + \frac{\ell}{\eta^2}\right)\mathrm{d}\lambda \lesssim_\varepsilon \eta^{1-\varepsilon} + \eta\log(1+\ell).
$$

Step 3.c: Case $l(Q_1) \ge 2R$.

As before in step 1.c, $|(\nabla\rho\theta)_{Q_1}| \le |(\nabla\rho\theta)_{Q_{2R}}|$, which can be further controlled by cubes that tile $\operatorname{supp}(\nabla\rho)$. Therefore, this case is bounded as in step 3.b.

Step 4: (4.2.37) holds; that is: $\|\nabla\rho\theta\|_{*,\,\text{dyadic}} \lesssim c(\eta)$

Here we have 3 cases to consider:

(1) $Q \subset Q_{2R}$.

(2) $Q \subset \mathbb{R}^n \setminus \operatorname{supp}(\nabla\rho)$.

(3) $Q_{2R} \subset Q$.

Case (2) is obvious. Case (3) reduces down to case (1) by step 1.c, the reflection and tiling of $\tilde{\phi}$, and considering $\operatorname{supp}(\nabla\rho)$.

Case (1): Using lemma 4.2.17 this reduces down to showing:

(a) $\dfrac{|\theta_Q|}{|Q|}\displaystyle\int_Q |\nabla\rho - (\nabla\rho)_Q| \lesssim c(\eta)$,

(b) $\dfrac{1}{|Q|}\displaystyle\int_Q |\nabla\rho(\theta - \theta_Q)| \lesssim c(\eta)$,

for Q dyadic and $Q \subset Q_{2R}$.

Step 4.a: (a) holds for Q dyadic and $Q \subset Q_{2R}$.

Step 4.a.i: Case $Q \subset Q_{2R}$ and $l(Q) \lesssim \frac{R\eta}{\ell}$.

By the naïve bounds in step 3.a $|\theta|_Q \lesssim \ell R$. If we use the mean value theorem for $\nabla\rho$ similar to step 2.b.i then

$$\frac{1}{|Q|}\int_Q |\nabla\rho - (\nabla\rho)_Q| \lesssim |\nabla^2\rho| l(Q) \lesssim \frac{l(Q)}{R^2}.$$

Therefore

$$\frac{|\theta_Q|}{|Q|}\int_Q |\nabla\rho - (\nabla\rho)_Q| \lesssim \ell R\frac{l(Q)}{R^2} \lesssim \eta.$$

Step 4.a.ii: Case $Q \subset Q_{2R}$ and $\frac{R\eta}{\ell} \lesssim l(Q) \leq 2R$.

Here we apply the same technique as step 3.b

$$\frac{|\theta_Q|}{|Q|}\int_Q |\nabla\rho - (\nabla\rho)_Q| \leq |\theta|_Q |\nabla\rho| \lesssim_\varepsilon \eta^{1-\varepsilon} + \eta\log(1+\ell).$$

Step 4.b: (b) holds for Q dyadic and $Q \subset Q_{2R}$.

$$\frac{1}{|Q|}\int_Q |\nabla\rho(\theta - \theta_Q)| \lesssim \frac{1}{R}\frac{1}{|Q|}\int_Q |\theta - \theta_Q|.$$

We split this into the now usual cases.

Step 4.b.i: Case $l(Q) \lesssim \frac{R\eta}{\ell}$.

By the intermediate and mean value theorems $|\tilde\phi - \tilde\phi_Q| \lesssim l(Q)\ell$ and $|x - x_Q| \lesssim l(Q)$ so

$$\frac{1}{R}\frac{1}{|Q|}\int_Q |\theta - \theta_Q| = \frac{1}{R}\frac{1}{|Q|}\int_Q |\tilde\phi - \tilde\phi_Q - x\cdot\nabla\tilde\phi_{Q_r} + (x\cdot\nabla\tilde\phi_{Q_r})_Q| \lesssim \frac{1}{R} l(Q)\ell \lesssim \eta.$$

Step 4.b.ii: Case $\frac{R\eta}{\ell} \lesssim l(Q) < 2R$.

$$\frac{1}{R}\frac{1}{|Q|}\int_Q |\theta - \theta_Q| \lesssim \frac{1}{R}|\theta|_Q$$

then applying the result from step 3.b gives

$$\frac{1}{|Q|}\int_Q |\nabla\rho(\theta - \theta_Q)| \lesssim_\varepsilon \eta^{1-\varepsilon} + \eta\log(1+\ell).$$

Therefore by lemma 4.2.14 we have shown $\nabla\Phi \in \mathrm{BMO}(\mathbb{R}^n)$ and the bound (4.2.45) holds. □

To finish proving theorem 4.2.13 we need to establish property (iv).

Lemma 4.2.19. *Let $\Phi : \mathbb{R}^{n-1}\times\mathbb{R} \to \mathbb{R}$ be defined in (4.2.35) with*

$$\sup_{\substack{Q_s = J_s\times I_s,\\ Q_s\subset Q_{8d},\, s\leq r_1}} \frac{1}{|Q_s|}\int_{Q_s}\int_{I_s} \frac{|\phi(x,t) - \phi(x,\tau)|^2}{|t-\tau|^2}\,\mathrm{d}\tau\,\mathrm{d}t\,\mathrm{d}x \leq \eta^2 \tag{4.2.46}$$

then Φ satisfies

$$\sup_{Q_s = J_s\times I_s} \frac{1}{|Q_s|}\int_{Q_s}\int_{I_s} \frac{|\Phi(x,t) - \Phi(x,\tau)|^2}{|t-\tau|^2}\,\mathrm{d}\tau\,\mathrm{d}t\,\mathrm{d}x \lesssim \eta^2. \tag{4.2.47}$$

Proof. Trivially since Φ is defined globally

$$\begin{aligned}
&\sup_{Q_s=J_s\times I_s}\frac{1}{|Q_s|}\int_{Q_s}\int_{I_s}\frac{|\Phi(x,t)-\Phi(x,\tau)|^2}{|t-\tau|^2}\,\mathrm{d}\tau\,\mathrm{d}t\,\mathrm{d}x\\
&\leq \sup_{Q_s=J_s\times I_s}\frac{1}{|Q_s|}\int_{Q_s}\int_{I_s}\frac{|\tilde\phi(x,t)-\tilde\phi(x,\tau)|^2}{|t-\tau|^2}\,\mathrm{d}\tau\,\mathrm{d}t\,\mathrm{d}x,
\end{aligned}$$

where we interpret the value of $\tilde\phi$ where it is undefined as 0, i.e. $\tilde\phi(x,t)=0$ when $(x,t)\notin\mathrm{supp}(\tilde\phi)$. It remains to establish

$$\begin{aligned}
&\sup_{I_s}\frac{1}{|I_s|}\int_{I_s}\int_{I_s}\frac{|\tilde\phi(x,t)-\tilde\phi(x,\tau)|^2}{|t-\tau|^2}\,\mathrm{d}\tau\,\mathrm{d}t\,\mathrm{d}x\\
&\lesssim \sup_{I_s\subset I_r}\frac{1}{|I_s|}\int_{I_s}\int_{I_s}\frac{|\phi(x,t)-\phi(x,\tau)|^2}{|t-\tau|^2}\,\mathrm{d}\tau\,\mathrm{d}t\,\mathrm{d}x
\end{aligned}\tag{4.2.48}$$

pointwise in x, where $Q_r=J_r\times I_r$ and is used to define Φ in (4.2.35). To simplify our notation we drop the dependence on the spatial variables in $\tilde\phi$ and ϕ. We also set $A:=I_s$. Recall from (4.2.34) that

$$\tilde\phi(t)=\begin{cases}\phi(t) & t\in[-r^2,r^2]+4kr^2,\\ \phi(2r^2-t) & t\in[r^2,3r^2]+4kr^2,\end{cases}$$

for $k\in\mathbb{Z}$. Let $I_k=[-r^2,r^2]+4kr^2$ and $J_k=[r^2,3r^2]+4kr^2$ be intervals in time for $k\in\mathbb{Z}$. We partition A into disjoint pieces $A=\cup_i I_i\cup_j J_j\cup A_1\cup A_2$, where A_1 and A_2 are the leftover pieces that do not contain either I_i or J_j.

If $A=A_1\cup A_2$ we may as well assume (by translation and reflection) that $A_1=[a,r^2]$ and $A_2=[r^2,b]$. Let τ',b' and A_2' be the images of τ,b and A_2 respectively under the map $\tau\mapsto 2r^2-\tau$. Without loss of generality we only consider the case $|A_1|>|A_2|$. Since $|t-\tau|=|t-r^2|+|\tau'-r^2|\geq|t-\tau'|$ we have for $t\in A_1$, $\tau\in A_2$

$$\begin{aligned}
&\int_{A_1}\int_{A_2}\frac{|\tilde\phi(t)-\tilde\phi(\tau)|^2}{|t-\tau|^2}\,\mathrm{d}\tau\,\mathrm{d}t=\int_a^{r^2}\int_{b'}^{r^2}\frac{|\phi(t)-\phi(\tau')|^2}{|t-(2t^2-\tau')|^2}\,\mathrm{d}\tau'\,\mathrm{d}t\\
&\leq\int_a^{r^2}\int_{b'}^{r^2}\frac{|\phi(t)-\phi(\tau')|^2}{|t-\tau'|^2}\,\mathrm{d}\tau'\,\mathrm{d}t\leq\int_{A_1}\int_{A_1}\frac{|\phi(t)-\phi(\tau')|^2}{|t-\tau'|^2}\,\mathrm{d}\tau'\,\mathrm{d}t.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\frac{1}{|A|}\int_A\int_A\frac{|\tilde\phi(t)-\tilde\phi(\tau)|^2}{|t-\tau|^2}\,\mathrm{d}\tau\,\mathrm{d}t\\
&=\frac{1}{|A|}\left(\int_{A_1}\int_{A_1}+2\int_{A_1}\int_{A_2}+\int_{A_2}\int_{A_2}\right)\frac{|\phi(t)-\phi(\tau')|^2}{|t-\tau'|^2}\,\mathrm{d}\tau'\,\mathrm{d}t\lesssim\eta^2.
\end{aligned}$$

In the general case when $A=\cup_{i\in\mathcal{I}}I_i\cup_{j\in\mathcal{J}}J_j\cup A_1\cup A_2$ we write the double integral over A in terms of integrals

$$\sum_{i,k\in\mathcal{I}}\int_{I_i}\int_{I_k}\frac{|\tilde\phi(t)-\tilde\phi(\tau)|^2}{|t-\tau|^2}\,\mathrm{d}\tau\,\mathrm{d}t,\qquad\sum_{i\in\mathcal{I},j\in\mathcal{J}}\int_{I_i}\int_{J_j}\frac{|\tilde\phi(t)-\tilde\phi(\tau)|^2}{|t-\tau|^2}\,\mathrm{d}\tau\,\mathrm{d}t$$

and integrals that involve sets A_1 or A_2 or both (those are handled similar to the earlier calculation). We ignore the case where we integrate over J_j and J_k since it's the same as the integrating over I_i and I_k case.

Dealing with the first case, if $i\neq k$, $t\in I_i$ and $\tau\in I_k$ then $|t-\tau|\sim r^2|i-k|$; if $i=k$ then

$|t-\tau| = |t'-\tau'|$. Therefore

$$\begin{aligned}
&\sum_{i,k\in\mathcal{I}} \int_{I_i}\int_{I_k} \frac{|\tilde{\phi}(t)-\tilde{\phi}(\tau)|^2}{|t-\tau|^2}\,\mathrm{d}\tau\,\mathrm{d}t \\
&\sim \sum_{i\in\mathcal{I}} \int_{I_0}\int_{I_0} \frac{|\phi(t)-\phi(\tau)|^2}{|t-\tau|^2}\,\mathrm{d}\tau\,\mathrm{d}t + \sum_{\substack{i,k\in\mathcal{I}\\ i\neq k}} \frac{1}{r^4|i-k|^2}\int_{I_0}\int_{I_0} |\phi(t)-\phi(\tau)|^2\,\mathrm{d}\tau\,\mathrm{d}t \\
&\leq \sum_{i\in\mathcal{I}} \int_{I_0}\int_{I_0} \frac{|\phi(t)-\phi(\tau)|^2}{|t-\tau|^2}\,\mathrm{d}\tau\,\mathrm{d}t + \sum_{\substack{i,k\in\mathcal{I}\\ i\neq k}} \frac{1}{|i-k|^2}\int_{I_0}\int_{I_0} \frac{|\phi(t)-\phi(\tau)|^2}{|t-\tau|^2}\,\mathrm{d}\tau\,\mathrm{d}t \\
&\lesssim |\mathcal{I}| \int_{I_0}\int_{I_0} \frac{|\phi(t)-\phi(\tau)|^2}{|t-\tau|^2}\,\mathrm{d}\tau\,\mathrm{d}t.
\end{aligned}$$

In the second case

$$\begin{aligned}
&\sum_{i\in\mathcal{I},j\in\mathcal{J}} \int_{I_i}\int_{J_j} \frac{|\tilde{\phi}(t)-\tilde{\phi}(\tau)|^2}{|t-\tau|^2}\,\mathrm{d}\tau\,\mathrm{d}t \\
&\lesssim \sum_{\substack{i\in\mathcal{I},j\in\mathcal{J}\\ |i-j|\leq 1}} \int_{I_0}\int_{I_0} \frac{|\phi(t)-\phi(\tau)|^2}{|t-\tau|^2} + \sum_{\substack{i\in\mathcal{I},j\in\mathcal{J}\\ |i-j|\geq 2}} \frac{1}{r^4(|i-j|-1)^2}\int_{I_0}\int_{I_0} |\phi(t)-\phi(\tau)|^2\,\mathrm{d}\tau\,\mathrm{d}t \\
&\lesssim (|\mathcal{I}|+|\mathcal{J}|) \int_{I_0}\int_{I_0} \frac{|\phi(t)-\phi(\tau)|^2}{|t-\tau|^2}\,\mathrm{d}\tau\,\mathrm{d}t.
\end{aligned}$$

Since $|A| \sim (|\mathcal{I}|+|\mathcal{J}|)|I_0|$ and I_0 is one of the time intervals considered in the supremum of (4.2.48)

$$\frac{1}{|A|}\int_A\int_A \frac{|\phi(t)-\phi(\tau)|^2}{|t-\tau|^2}\,\mathrm{d}\tau\,\mathrm{d}t \sim \frac{1}{|I_0|}\int_{I_0}\int_{I_0} \frac{|\phi(t)-\phi(\tau)|^2}{|t-\tau|^2}\,\mathrm{d}\tau\,\mathrm{d}t \lesssim \eta^2.$$

Therefore we have proved theorem 4.2.13. □

We are now ready to define the class of parabolic domains on which we work.

Definition 4.2.20 (Admissible parabolic domains)**.** *We say $\Omega \subset \mathbb{R}^n \times \mathbb{R}$ is an admissible domain with character (ℓ, η, N, d) if there exists a positive scale r_1, and a modulus of continuity δ such that for any time $\tau \in \mathbb{R}$ there are at most N ℓ-cylinders $\{\mathbb{Z}_j\}_{j=1}^N$ of diameter d satisfying the following conditions:*

(1) $\partial\Omega \cap \{|t-\tau| \leq d^2\} = \bigcup_j (\mathbb{Z}_j \cap \partial\Omega)$.

(2) In the coordinate system (x_0, x, t) of the ℓ-cylinder $\mathbb{Z}_j$

$$\mathbb{Z}_j \cap \Omega \supset \left\{(x_0,x,t) \in \Omega : |x| < d, |t| < d^2, \delta(x_0,x,t) \leq d/2\right\}.$$

(3) $8\mathbb{Z}_j \cap \partial\Omega$ is the graph $\{x_0 = \phi_j(x,t)\}$ of a function $\phi_j : Q_{8d} \to \mathbb{R}$, with $Q_{8d} \subset \mathbb{R}^{n-1}\times\mathbb{R}$, such that

$$|\phi_j(x,t)-\phi_j(y,s)| \leq \ell\left(|x-y| + |t-s|^{1/2}\right) \quad \textit{and} \quad \phi_j(0,0) = 0. \tag{4.2.49}$$

(4)

$$d(\nabla\phi_j, C_\delta) \leq \eta \tag{4.2.50}$$

and

$$\sup_{\substack{Q_s = J_s\times I_s\\ Q_s\subset Q_{8d},\, s\leq r_1}} \frac{1}{|Q_s|}\int_{Q_s}\int_{I_s} \frac{|\phi_j(x,t)-\phi_j(x,\tau)|^2}{|t-\tau|^2}\,\mathrm{d}\tau\,\mathrm{d}t\,\mathrm{d}x \leq \eta^2. \tag{4.2.51}$$

We say that Ω *is a* VMO*-type domain if* η *in the character* (ℓ, η, N, d) *can be taken arbitrarily small (at the expense of a potentially smaller* d *and* r_1*, and larger* N*).*

Remark 4.2.21. When (4.2.50) holds for small or vanishing η it follows that for a fixed time τ the normal ν to the fixed-time spatial domain $\Omega_\tau = \Omega \cap \{t = \tau\}$ can be written in local coordinates as

$$\nu = \frac{1}{|(-1, \nabla\phi_j)|}(-1, \nabla\phi_j)$$

and hence $d(\nu, \mathrm{VMO}) \lesssim \eta$. Therefore Ω_τ is similar to the domains considered in the papers [MMS09] and [HMT15] which have dealt with the elliptic problems on domains with normal in or near VMO.

Remark 4.2.22. Theorems 4.1.3 to 4.1.5 from [DH18; DPP17] can be trivially extended to admissible domains defined in definition 4.2.20 above, with η replacing ℓ in the assumptions of the theorems as appropriate.

Corollary 4.2.23. *Let* Ω *be defined as in definition 4.2.20 by a family of functions* $\{\phi_j\}$*,* $\phi_j : Q_{8d} \to \mathbb{R}$*. There exists an extended family* $\{\Phi_j\}$*,* $\Phi_j : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}$*, such that*

(i) $\left\{ \Phi_j|_{Q_{8r}} \right\}$ *still describes* Ω*, as in definition 4.2.20, but with character* $(\ell, \eta, \tilde{N}, r)$ *instead of* (ℓ, η, N, d)*, where* $\tilde{N} \geq N$ *and* $r \leq r_1 \leq d$ *is from theorem 4.2.13.*

(ii) $\|\nabla\Phi_j\|_* \lesssim_\varepsilon \eta^{1-\varepsilon} + \eta\ell$.

(iii) $\|\mathbb{D}\Phi_j\|_* \lesssim_\varepsilon \eta^{1-\varepsilon} + \eta\ell$.

Proof. This follows from theorem 4.2.13 and by tiling the support of each ϕ_j into parabolic cubes of size $8r$ with enough overlap. ☐

Corollary 4.2.24. *If* Ω *is a* VMO*-type domain then we may take* η *arbitrarily small in corollary 4.2.23 by reducing* r*.*

Remark 4.2.25. From definitions 4.2.3 and 4.2.20, corollary 4.2.23, and theorems 4.2.5 and 4.2.7 we can easily see that the class of domains described by Lewis-Murray cylinders (definition 4.2.3) and admissible domains (definition 4.2.20) coincide. However, there are a few important advantages to admissible domains:

(i) The definition of admissible domains is truly local — as we expected from the properties of solutions to parabolic PDE (notably time irreversibility and exponential decay). This also means that it is easier to verify if a given domain is an admissible domain.

(ii) For admissible domains, we have much more nuanced control over the norms of the graphs and only need the BMO norms to be small (or close to VMO) locally instead of global conditions.

(iii) For a given ℓ and η the class of admissible domains is larger than the class of Lewis-Murray cylinders. Hence for a given p we are able to infer $(D)_p$ for a larger class of domains.

4.2.4 Pullback Transformation and Carleson Condition

We now study the pullback mapping of Dahlberg-Kenig-Stein [Dah86] on the upper half-space U $\rho : U \to \Omega$ in the setting of parabolic equations and how this relates to the Carleson condition on the coefficients (4.1.6) to (4.1.8). The following motivation comes from [HL01].

For simplicity assume $\tilde{u}$ is a solution to the heat equation ($\tilde{u}_t = \Delta\tilde{u}$) in

$$\Omega = \{(x_0, x, t) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R} : x_0 > \phi(x, t)\} \tag{4.2.52}$$

where $\phi(x, t) : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}$ and satisfies conditions (3) and (4) of definition 4.2.20. Suppose we natively assume the same pullback mapping as we did for the boundary of $\mathrm{Lip}(1, 1/2)$ domains in definition 2.1.5

$$\tilde{\rho}(x_0, x, t) = (x_0 + \phi(x, t), x, t).$$

So $\tilde{\rho}$ flattens the boundary. Then $\tilde{\rho}$ maps U onto Ω and ∂U onto $\partial\Omega$ bijectively. Furthermore, $u = \tilde{u} \circ \tilde{\rho}$ is a parabolic PDE of the form (4.1.1) where

$$B \cdot \nabla u = \phi_t(x,t) u_{x_0}(x_0, x, t),$$

for $(x_0, x, t) \in U$; and A is independent of x_0 and has the usual ellipticity properties. However, ϕ doesn't have a time derivative and so ϕ_t may not exist at all.

To overcome this we introduce the Dahlberg-Kenig-Stein mapping (c.f. [HL96]) which smooths out $\tilde{\rho}$ whilst still maintaining the same bijective properties we desire. We swap the x_0 independence of A for a Carleson measure condition. We define a *parabolic approximation to the identity* P to be an even non-negative function $P(x,t) \in C_0^\infty(Q_1(0,0))$ for $(x,t) \in \mathbb{R}^{n-1} \times \mathbb{R}$ with $\int P(x,t)\, \mathrm{d}x\, \mathrm{d}t = 1$ and as usual let

$$P_\lambda(x,t) := \lambda^{-(n+1)} P\left(\frac{x}{\lambda}, \frac{t}{\lambda^2}\right).$$

Let $P_\lambda \phi$ be the convolution operator

$$P_\lambda \phi(x,t) := \int_{\mathbb{R}^{n-1} \times \mathbb{R}} P_\lambda(x-y, t-s) \phi(y,s)\, \mathrm{d}y\, \mathrm{d}s$$

then P satisfies

$$\lim_{(y_0,y,s) \to (0,x,t)} P_{\gamma y_0} \phi(y,s) = \phi(x,t),$$

for small enough constants $\gamma > 0$. Let

$$\rho(x_0, x, t) = (x_0 + P_{\gamma x_0} \phi(x,t), x, t) \tag{4.2.53}$$

then ρ maps the upper half-space into Ω and extends continuously to $\rho : \overline{U} \to \overline{\Omega}$. This transformation allows us to consider the L^p solvability of the PDE (4.1.1) in the upper half-space instead of in the original domain Ω. We note that if $\tilde{u}$ is a solution to the heat equation then $u = \tilde{u} \circ \tilde{\rho}$ is a parabolic PDE of the form (4.1.1) where A is now dependent on x_0 and has the usual ellipticity properties. However, even though we lose x_0 independence we do gain a Carleson measure condition which we describe in due course. In addition, by [LM95, Chapter 3] the parabolic measure on $\partial\Omega$ defined with respect to this pullback PDE (coming from the heat equation) is an $A_\infty(\mathrm{d}x)$ weight with respect to the usual Lebesgue measure on ∂U. Furthermore, the usual surface measure on ∂U is comparable with the measure σ defined by (2.1.9) on $\partial\Omega$.

Suppose u solves a PDE of the form (4.1.1). If we let $v = u \circ \rho$ and $f^v = f \circ \rho$ then (4.1.1) transforms to a new PDE for the variable v

$$\begin{cases} v_t = \operatorname{div}(A^v \nabla v) + B^v \cdot \nabla v & \text{in } U, \\ v = f^v & \text{on } \partial U, \end{cases} \tag{4.2.54}$$

where $A^v = [a_{ij}^v(X,t)]$, $B^v = [b_i^v(X,t)]$ are $(n \times n)$ and $(1 \times n)$ matrices.

The precise relations between the original coefficients, A and B, and the new coefficients, A^v and B^v, are detailed in [Riv14, pp. 448]. We note that if the constant $\gamma > 0$ is chosen small enough then the coefficients $a_{ij}^v, b_i^v : U \to \mathbb{R}$ are Lebesgue measurable and A^v satisfies the standard uniform ellipticity condition with constants λ^v and Λ^v since the original matrix A did.

We now return back to the pullback transformation and investigate the Carleson condition on the coefficients A and B. The following result comes directly from a careful reading of the proof of Lemma 2.8 in [HL96] combined with theorems 4.2.5 and 4.2.7 along with the harmonic analysis results in section 2.5. Hofmann and Lewis [HL96] assume the stronger condition of $\phi \in \mathrm{Lip}(1, 1/2)$ and $D_{1/2}^t \phi \in \mathrm{BMO}(\mathbb{R}^n)$, see theorems 4.2.5 and 4.2.7 and the results following in section 4.2.2 for a comparison of these conditions. The main difference is that their result requires $\nabla\phi \in L^\infty$ whereas we only require $R_j \partial_j \phi \in \mathrm{BMO}$ for all $1 \le j \le n$; $\nabla\phi \in \mathrm{BMO}$ would imply this.

Lemma 4.2.26. *Let σ and θ be non-negative integers, $\alpha = (\alpha_1, \dots, \alpha_{n-1})$ a multi-index with*

$l = \sigma + |\alpha| + \theta$, d a scale and fix γ. If $\phi : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}$ satisfies

$$\|\mathbb{D}\phi\|_* \le \eta \tag{4.2.55}$$

then the measure ν defined at (x_0, x, t) by

$$\mathrm{d}\nu = \left(\frac{\partial^l P_{\gamma x_0} \phi}{\partial x_0^\sigma \partial x^\alpha \partial t^\theta} \right)^2 x_0^{2l+2\theta-3} \,\mathrm{d}x \,\mathrm{d}t \,\mathrm{d}x_0 \tag{4.2.56}$$

is a Carleson measure on cubes of diameter $\le d/4$ whenever either $\sigma + \theta \ge 1$ or $|\alpha| \ge 2$ with

$$\nu\left(T(Q_r)\right) \lesssim \eta |Q_r|, \tag{4.2.57}$$

where $r \le d/4$. If $l \ge 1$ then for (x_0, x, t) with $x_0 \le d/4$

$$\left| \frac{\partial^l P_{\gamma x_0} \phi}{\partial x_0^\sigma \partial x^\alpha \partial t^\theta} \right| \lesssim \eta x_0^{1-l-\theta}, \tag{4.2.58}$$

where the implicit constants depend on d, l, n.

Moreover, if either $\sigma + \theta \ge 1$ or $|\alpha| \ge 2$ then

$$\lim_{(x_0,x,t)\to(0,y,s)} \left(x_0^{(l+\theta-1)} \frac{\partial^l P_{\gamma x_0} \phi}{\partial x_0^\sigma \partial x^\alpha \partial t^\theta} \right) = 0 \quad \textit{for a.e. } (y,s) \in \mathbb{R}^{n-1} \times \mathbb{R}. \tag{4.2.59}$$

Proof. We only sketch a brief outline of this proof from [HL96] and note where we can improve their estimate if we only use the weaker assumption (4.2.55). We consider two cases of the indices:

$$\frac{\partial P_{\gamma x_0} \phi}{\partial t} \quad \text{and} \quad \frac{\partial P_{\gamma x_0} \phi}{\partial x_0};$$

with the other cases being similar.

First the time derivative, where $\theta = l = 1$ and $\sigma = |\alpha| = 0$. From the definitions of $\mathbb{D}$, $\mathbb{D}_n$ and R_j in (4.2.5) to (4.2.8) we have

$$\begin{aligned} \frac{\partial P_{\gamma x_0} \phi}{\partial t} &= -i(\mathbb{D} P_{\gamma x_0}) * \mathbb{D}_n \phi \\ &= (\gamma x_0)^{-1} \tilde{Q}_{\gamma x_0} * \mathbb{D}_n \phi, \end{aligned} \tag{4.2.60}$$

where $\tilde{Q}_{x_0}$ is a generic approximation to the zero operator $\tilde{Q}_{x_0} \in C^\infty(\mathbb{R}^n)$. That is $\tilde{Q}_{x_0}$ satisfies $\int_{\mathbb{R}^n} \tilde{Q}_{x_0} = 0$ but might not be compactly supported. Instead $\tilde{Q}_{x_0}$ satisfies the standard parabolic kernel estimates with $\delta = 1$ from (2.5.13) — recall that those are:

$$\begin{aligned} |\tilde{Q}_{x_0}(z)| &\lesssim \frac{1}{\|z\|^{n+1}}, \\ |\tilde{Q}_{x_0}(z) - \tilde{Q}_{x_0}(v)| &\lesssim \frac{\|z-v\|}{\|z\|^{n+2}} \quad \text{if} \quad \|z\| > 2\|z - v\|. \end{aligned} \tag{2.5.13}$$

Since $\mathbb{D}_n = R_n \mathbb{D}$ then, by corollary 2.5.27 and theorem 2.5.14, equation (4.2.56) is a Carleson measure with norm controlled by η. The L^∞ norm estimate (4.2.58) follows from corollary 2.5.27, theorem 2.5.14, and lemma 2.5.17.

Now for the partial derivative in the x_0 direction, when $\sigma = l = 1$ and $\theta = |\alpha| = 0$. We begin by observing that $\frac{\partial P_{\gamma x_0} \phi}{\partial x_0} = x_0^{-1} Q^{(0)}_{\gamma x_0}$, where $Q^{(0)}_{x_0}$ denotes an approximation to the zero operator which has first spatial moments equal to zero, see (4.2.61) below. That is a compactly supported smooth kernel satisfying $\int Q^{(0)}_{x_0} = 0$, (2.5.13) and the following moment condition for $1 \le j \le n-1$

$$\int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} x_j Q^{(0)}_{x_0}(x,t) \,\mathrm{d}x \,\mathrm{d}t = 0. \tag{4.2.61}$$

By [Hof97, Lemma 1 in §2] $x_0^{-1}\mathbb{D}^{-1}Q_{x_0}^{(0)} = \tilde{Q}_{x_0}$, where $\tilde{Q}_{x_0}$ is as above. Therefore

$$\begin{aligned} \frac{\partial P_{\gamma x_0}}{\partial x_0} &= \left(\mathbb{D}^{-1}\frac{\partial P_{\gamma x_0}\phi}{\partial x_0}\right) * \mathbb{D}\phi \\ &= \gamma\tilde{Q}_{\gamma x_0} * \mathbb{D}\phi \end{aligned} \tag{4.2.62}$$

and we proceed as before to obtain (4.2.57) and (4.2.58).

The proof of (4.2.59) only uses (4.2.57) and (4.2.58). Since we obtain (4.2.57) and (4.2.58) then (4.2.59) can be proved by the argument outlined in [HL96]. $\square$

Applying the pullback transformation

The drift term B^v from the pullback transformation in (4.2.54) includes the term

$$\frac{\partial P_{\gamma x_0}\phi}{\partial t}u_{x_0}.$$

From lemma 4.2.26 with $\sigma = |\alpha| = 0$, $\theta = 1$, we see that

$$x_0\left[\frac{\partial P_{\gamma x_0}\phi(x,t)}{\partial t}\right]^2 \mathrm{d}X\,\mathrm{d}t$$

is a Carleson measure in U. Thus it is natural to expect that

$$\mathrm{d}\mu_1(X,t) = x_0|B^v|^2(X,t)\,\mathrm{d}X\,\mathrm{d}t \tag{4.2.63}$$

is a Carleson measure in U and B^v satisfies

$$x_0|B^v|(X,t) \leq \Lambda_B < \|\mu_1\|_C^{1/2}. \tag{4.2.64}$$

Indeed, this is the case provided the original vector B satisfies the assumption that

$$\mathrm{d}\mu(X,t) = \delta(X,t)\left[\sup_{B_{\delta(X,t)/2}(X,t)}|B|\right]^2 \mathrm{d}X\,\mathrm{d}t \tag{4.2.65}$$

is a Carleson measure in Ω. Here $\|\mu_1\|_C$ depends on η and the Carleson norm of (4.2.65).

Similarly, for the matrix A^v if we apply lemma 4.2.26 and use the calculations in [Riv14, §6] then

$$\mathrm{d}\mu_2(X,t) = (x_0|\nabla A^v|^2 + x_0^3|A_t^v|^2)(X,t)\,\mathrm{d}X\,\mathrm{d}t \tag{4.2.66}$$

is a Carleson measure in U and A^v satisfies

$$(x_0|\nabla A^v| + x_0^2|A_t^v|)(X,t) \leq \|\mu_2\|_C^{1/2} \tag{4.2.67}$$

for almost everywhere $(X,t) \in U$ provided the original matrix A satisfies that

$$\mathrm{d}\mu(X,t) = \left(\delta(X,t)\left[\sup_{B_{\delta(X,t)/2}(X,t)}|\nabla A|\right]^2 + \delta(X,t)^3\left[\sup_{B_{\delta(X,t)/2}(X,t)}|\partial_t A|\right]^2\right)\mathrm{d}X\,\mathrm{d}t \tag{4.2.68}$$

is a Carleson measure in Ω.

We note that if both $\|\mu\|_{C,r}$ and η are small then so too are the Carleson norms $\|\mu_1\|_{C,r}$ and $\|\mu_2\|_{C,r}$ of the matrix A^v and vector B^v, at least if we restrict ourselves to small Carleson regions $r \leq d$; this comes from theorem 4.2.13 and corollaries 4.2.23 and 4.2.24. By lemma 4.2.26 we see that $\|\mu_1\|_{C,r}$ and $\|\mu_2\|_{C,r}$ only depend on η and $\|\mu\|_{C,r}$ on Carleson regions of size $r \leq d$. In particular they are small if both η and $\|\mu\|_{C,r}$ are small. It further follows by corollary 4.2.24 that we can make $\|\mu_1\|_{C,r}$ and $\|\mu_2\|_{C,r}$ as small as we like if μ is a vanishing Carleson norm and the domain Ω is of VMO-type.

Observe that condition (4.2.68) is slightly stronger than (4.1.6), which we claimed to assume in theorem 4.1.6. We replace condition (4.2.68) by the weaker condition (4.1.6) later via the perturbation result of [Swe98; Nys97] in theorem 2.4.33.

Definition 4.2.27. *We define $\rho_j : U \to 8\mathbb{Z}_j$ as in* (4.2.53) *to be the local pullback mapping in $8\mathbb{Z}_j$ associated to the function Φ_j in theorem 4.2.13, the extension of ϕ_j from definition 4.2.20.*

Remark 4.2.28. By [BZ17] (and similar to its adaptation to the setting of Lewis-Murray cylinders in [DH18, §2.3]) one may construct a 'proper generalised distance' globally. This can be done by first constructing the signed distance to the boundary function d in [BZ17] globally and then using the regularisation technique. We may then use the result of [BZ17, Theorem 5.1] to show there exists a domain Ω^ε of class C^∞ and a homeomorphism $f^\varepsilon : \overline{\Omega} \to \overline{\Omega^\varepsilon}$ such that $f^\varepsilon(\partial\Omega) = \partial\Omega^\varepsilon$ and $f^\varepsilon : \Omega \to \Omega^\varepsilon$ is a C^∞ diffeomorphism.

4.3 Basic Results for Parabolic Time-varying Domains with Drift Terms

All of the interior results developed in chapter 2 hold in Lewis-Murray cylinders and admissible domains with drift terms; these are subclasses of $\mathrm{Lip}(1, 1/2)$ domains. A few results that we may want to use, namely the boundary Hölder continuity, the double property of the parabolic measure, and the existence of the Green's function lemmas 2.2.9, 2.3.4 and 2.3.6 are stated for $B \equiv 0$ in chapter 2. Therefore we state the extended results for parabolic PDE with drift terms in time-varying domains satisfying the Lewis-Murray condition. First an example of why we need to introduce a smallness condition on the drift term B, even in the elliptic setting.

Remark 4.3.1 ([HL01])**.** Let

$$B = \left(\frac{1}{x_0}\left[1 + \frac{2}{\log x_0}\right], 0, \dots, 0\right)$$

then $u(X, t) = -(\log x_0)^{-1}$ solves $\Delta u + B \cdot \nabla u = 0$ for $0 < x_0 < 1/2$. Therefore we need some smallness condition, like (1.0.3) for small K, for the boundary Hölder continuity to be valid.

Proposition 4.3.2 ([HL01])**.** *Let Ω be a Lewis-Murray type domain or an admissible domain, either definition 4.2.3 or 4.2.20, with $\eta < 1$ sufficiently small. If $A, B \in C^\infty(\overline{\Omega})$, A is bounded and elliptic, and B satisfies* (1.0.3) *for $K < 1$ sufficiently small, that is*

$$\delta(X, t)|B(X, t)| \leq K,$$

then the boundary Hölder continuity, and the existence and properties of the Green's function hold (lemmas 2.2.9 and 2.3.6).

The smoothness assumption on A and B can be dropped for the doubling properties of the parabolic measure, lemma 2.3.4, by an approximation argument, c.f. [HL01].

Proposition 4.3.3 ([HL01])**.** *Let Ω be a Lewis-Murray type domain or an admissible domain, either definition 4.2.3 or 4.2.20, with $\eta < 1$ sufficiently small. If A is bounded and elliptic, and B satisfies* (1.0.3) *for $K < 1$ sufficiently small then the parabolic doubling and corkscrew point lemma holds, lemma 2.3.4.*

Remark 4.3.4. It is an open question that if instead of assuming (1.0.3) for propositions 4.3.2 and 4.3.3 can we instead assume smallness in the Carleson condition either by (4.1.6) or (4.1.7)?

4.4 The p-adapted Square and Area Functions

The following p-adapted square function was introduced in [DPP07] and has been modified appropriately for the parabolic setting. It is used to control the spatial derivatives of the solution. When $p = 2$ it is equivalent to the usual square function and when $p < 2$ we use the convention that the expression $|\nabla u|^2 |u|^{p-2}$ is zero whenever ∇u vanishes.

Definition 4.4.1 (p-adapted square function)**.** *For a function $u : \Omega \to \mathbb{R}$ the p-adapted square function $S_{p,a}(u) : \partial\Omega \to \mathbb{R}$ and its truncated version at a height r are defined as*

$$\begin{aligned} S_{p,a}(u)(Y,s) &= \left(\int_{\Gamma_a(Y,s)} |\nabla u(X,t)|^2 |u(X,t)|^{p-2} \delta(X,t)^{-n} \,\mathrm{d}X\,\mathrm{d}t \right)^{1/p}, \\ S^r_{p,a}(u)(Y,s) &= \left(\int_{\Gamma^r_a(Y,s)} |\nabla u(X,t)|^2 |u(X,t)|^{p-2} \delta(X,t)^{-n} \,\mathrm{d}X\,\mathrm{d}t \right)^{1/p}. \end{aligned} \tag{4.4.1}$$

By applying Fubini we have

$$\|S_{p,a}(u)\|^p_{L^p(\partial U)} \sim \int_U |\nabla u|^2 |u|^{p-2} x_0 \,\mathrm{d}x_0\,\mathrm{d}x\,\mathrm{d}t. \tag{4.4.2}$$

It is not know a priori if these integrals are locally integrable even for $p > 2$. However, theorem 4.4.3 and remark 4.5.3 show that these expressions makes sense and are finite for solutions to (4.1.1)*.*

We also need a p-adapted version of an object called the area function which was introduced in [DH18] and is used to control the solution in the time variable. Again when $p = 2$ this is just the area function of [DH18].

Definition 4.4.2 (p-adapted area function)**.** *For a function $u : \Omega \to \mathbb{R}$ the p-adapted area function $A_{p,a}(u) : \partial\Omega \to \mathbb{R}$ and its truncated version at a height r are defined as*

$$\begin{aligned} A_{p,a}(u)(Y,s) &= \left(\int_{\Gamma_a(Y,s)} |u_t|^2 |u(X,t)|^{p-2} \delta(X,t)^{2-n} \,\mathrm{d}X\,\mathrm{d}t \right)^{1/p}, \\ A^r_{p,a}(u)(Y,s) &= \left(\int_{\Gamma^r_a(Y,s)} |u_t|^2 |u(X,t)|^{p-2} \delta(X,t)^{2-n} \,\mathrm{d}X\,\mathrm{d}t \right)^{1/p}. \end{aligned} \tag{4.4.3}$$

By Fubini

$$\|A_{p,a}(u)\|^p_{L^p(\partial U)} \sim \int_U |u_t|^2 |u|^{p-2} x_0^3 \,\mathrm{d}x_0\,\mathrm{d}x\,\mathrm{d}t. \tag{4.4.4}$$

As before, it is not known a priori if these expressions are finite for solutions to (4.1.1) *but in lemma 4.4.10 we establish control of $A_{p,a}$ by $S_{p,2a}$ and use the finiteness of $S_{p,a}$ from theorem 4.4.3 and remark 4.5.3.*

4.4.1 Improved Regularity for the p-adapted Square and Area Functions

Here we extend the recent work of Dindoš and Pipher [DP16] from complex coefficient elliptic equations to the real parabolic setting. The goal is to obtain a improved regularity result for weak solutions of (4.1.1) implying that $|\nabla u|^2 |u|^{p-2}$ belongs to $L^1_{loc}(\Omega)$ when $1 < p < 2$. Having this it follows via remark 4.5.3 that the p-adapted square function $S_{p,a}$ is well defined at almost every boundary point. Furthermore we prove a p-adapted version of a Caccioppoli inequality for the second gradient in proposition 4.4.9. This allows us to control the p-adapted area function by the p-adapted square function.

Theorem 4.4.3 (c.f. [DP16, Theorem 1.1])**.** *Suppose $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ is a weak solution to $Lu = 0$, where $Lu = \mathrm{div}(A\nabla u) + B\nabla u - u_t$, A is bounded and elliptic, B is locally bounded and satisfies*

$$\delta(X,t)|B(X,t)| \le K \tag{4.4.5}$$

for some uniform constant $K > 0$. For any parabolic ball $B_{4r}(X,t) \subset \Omega$, any $\varepsilon > 0$ and any

$p, q \in (1, \infty)$ *we have the following improvement in regularity*

$$\left(\fint_{B_r(X,t)} |u|^p \right)^{1/p} \leq C_\varepsilon \left(\fint_{B_{2r}(X,t)} |u|^q \right)^{1/q} + \varepsilon \left(\fint_{B_{2r}(X,t)} |u|^2 \right)^{1/2}. \tag{4.4.6}$$

Here the constant C_ε only depends on p, q, ε, n, λ, Λ, and K but not on u, (X, t) or r. In addition, for all $1 < p < \infty$

$$r^2 \fint_{B_r(X,t)} |\nabla u|^2 |u|^{p-2} \leq C_\varepsilon \fint_{B_{2r}(X,t)} |u|^p + \varepsilon \left(\fint_{B_{2r}(X,t)} |u|^2 \right)^{p/2}, \tag{4.4.7}$$

where again the constant only depends on ε, p, n, the ellipticity constants of A, and K. This shows that $|u|^{(p-2)/2} \nabla u \in L^2_{\text{loc}}(\Omega)$.

Remark 4.4.4. If $q \geq 2$ in (4.4.6) or if $p \geq 2$ in (4.4.7) then one can take $\varepsilon = 0$ because the L^2 averages can be controlled by the first term on the right hand side of these inequalities.

The case of $p \geq 2$ below follows from the Caccioppoli inequality, lemma 2.2.1.

Lemma 4.4.5 ([DP16, Lemma 2.6])**.** *Let u be a weak solution to $Lu = 0$ in Ω for A elliptic and bounded, and B bounded satisfying* (4.4.5)*. For any $p > 2$ and any ball $B_r(X, t)$ with $r < \delta(X, t)/4$*

$$\int_{B_r(X,t)} |\nabla u|^2 |u|^{p-2} \lesssim r^{-2} \int_{B_{2r}(X,t)} |u|^p \tag{4.4.8}$$

and

$$\left(\fint_{B_r(X,t)} |u|^p \right)^{1/p} \lesssim \left(\fint_{B_{2r}(X,t)} |u|^2 \right)^{1/2}. \tag{4.4.9}$$

The implied constant depends on n, p, λ, Λ and K. In particular, $|u|^{(p-2)/2} u \in W^{1,2}_{\text{loc}}(\Omega)$.

Dindoš and Pipher [DP16] use a condition called p-ellipticity however since our matrix A is real and elliptic then it is p-elliptic for all $1 < p < \infty$.

Lemma 4.4.6 ([DP16, Theorem 2.4])**.** *Assume that A is elliptic then there exists $\lambda' = \lambda'(\lambda, \Lambda)$ such that for any non-negative, bounded and measurable function χ, and any u such that $|u|^{(p-2)/2} u \in W^{1,2}_{\text{loc}}(\Omega)$ we have*

$$\int_\Omega A \nabla u \cdot \nabla \left(|u|^{p-2} u \right) \chi \geq \lambda' \int_\Omega |u|^{p-2} |\nabla u|^2 \chi. \tag{4.4.10}$$

Lemma 4.4.7 ([DP16, Lemma 2.5])**.** *For all $p > 1$ and (X, t) such that $u(X, t) \neq 0$*

$$\left| \nabla \left(|u(X,t)|^{p/2-1} u(X,t) \right) \right|^2 \sim |u(X,t)|^{p-2} |\nabla u(X,t)|^2. \tag{4.4.11}$$

We shall establish the following lemma for the $1 < p < 2$ case which concludes the proof of theorem 4.4.3.

Lemma 4.4.8 (c.f. [DP16, Lemma 2.7])**.** *Let u be a weak solution to $Lu = 0$ in Ω for A elliptic and bounded, and B bounded satisfying* (4.4.5)*. For any $p < 2$, any ball $B_r(X, t)$ with $r < \delta(X, t)/4$, and for any $\varepsilon > 0$*

$$r^2 \int_{B_r(X,t)} |\nabla u|^2 |u|^{p-2} \leq C_\varepsilon \fint_{B_{2r}(X,t)} |u|^p + \varepsilon \left(\fint_{B_{2r}(X,t)} |u|^2 \right)^{p/2} \tag{4.4.12}$$

and

$$\left(\fint_{B_r(X,t)} |u|^2 \right)^{1/2} \leq C_\varepsilon \left(\fint_{B_{2r}(X,t)} |u|^p \right)^{1/p} + \varepsilon \left(\fint_{B_{2r}(X,t)} |u|^2 \right)^{1/2}, \tag{4.4.13}$$

where the constants only depend on n, ε, λ, Λ and K. In particular, $|u|^{(p-2)/2}\nabla u \in L^2_{\mathrm{loc}}(\Omega)$.

Proof. We start by assuming that A and B are smooth then the solution u to $Lu = 0$ is smooth. We prove the above inequalities with constants that do not depend on the smoothness of A or B and then remove the smoothness assumption at the end of the proof via an approximation argument in [HL01]. To simplify notation we suppress the argument of the ball $B_r(X,t)$.

Let

$$\rho_\delta(s) = \begin{cases} \delta^{(p-2)/2} & 0 \le s \le \delta, \\ s^{(p-2)/2} & s > \delta. \end{cases} \tag{4.4.14}$$

The choice of the cut off function ρ_δ in this proof is inspired by [Lan99, p. 311; CM05, p. 1088]. We multiply $Lu = 0$ by $\rho_\delta^2(|u|)u$ and integrate by parts to obtain

$$\begin{aligned} \int_{B_r} \nabla\left(\rho_\delta^2(|u|)u\right) A\nabla u = \int_{B_r} \rho_\delta^2(|u|)uu_t + \int_{B_r} \rho_\delta^2(|u|)B\cdot\nabla u \\ + \int_{\partial B_r} (\rho_\delta^2(|u|))\nu \cdot A\nabla u \,\mathrm{d}\sigma(y,s), \end{aligned} \tag{4.4.15}$$

where ν is the outer unit normal to B_r. Consider $E_\delta = \{u > \delta\}$ then the left hand side of (4.4.15) is

$$\int_{B_r} \nabla\left(\rho_\delta^2(|u|)u\right) A\nabla u = \delta^{p-2}\int_{B_r\setminus E_\delta} \nabla u\cdot A\nabla u + \int_{B_r\cap E_\delta} A\nabla u\cdot\nabla\left(|u|^{p-2}u\right) \tag{4.4.16}$$

and by ellipticity of A on the open set $B_r \cap E_\delta$ we have for some $\lambda' > 0$

$$\lambda'\int_{B_r\cap E_\delta} |u|^{p-2}|\nabla u|^2 \le \int_{B_r\cap E_\delta} A\nabla u\cdot\nabla\left(|u|^{p-2}u\right). \tag{4.4.17}$$

Our strategy is to let $\delta \to 0$ and show that all the integrals involving $B_r \setminus E_\delta$ tend to 0.

First, we use the following result from [Lan99]. They proved if $u \in C^2\left(\overline{B_r}\right)$ and $u = 0$ on ∂B_r then for $q > -1$

$$\lim_{\delta\to 0} \delta^q \int_{B_r\setminus E_\delta} |\nabla u|^2 = 0. \tag{4.4.18}$$

To deal with the boundary integral in (4.4.15) we note that equations (4.4.15) to (4.4.17) remain valid for any enlarged ball $B_{\alpha r}$ for $1 \le \alpha \le 5/4$. We write (4.4.15) for every $B_{\alpha r}$ and then average in α over the interval $[1, 5/4]$. The last term in (4.4.15) then turns into a solid integral over $B_{5r/4} \setminus B_r$. Therefore

$$\begin{aligned} \lambda'\int_{B_r\cap E_\delta} |u|^{p-2}|\nabla u|^2 &\le \sup_{\alpha\in[1,5/4]}\left|\int_{B_{\alpha r}} \rho_\delta^2\left(|u|\right)uu_t\right| + \sup_{\alpha\in[1,5/4]}\left|\int_{B_{\alpha r}} \rho_\delta^2\left(|u|\right)uB\cdot\nabla u\right| \\ &\quad + \left|r^{-1}\int_{B_{5\alpha r/4}\setminus B_r} \rho_\delta^2\left(|u|\right)u\nu\cdot A\nabla u\right| + o(1) \\ &= I + II + III + o(1), \end{aligned}$$

where $o(1)$ contains the integral over $B_{\alpha r} \setminus E_\delta$ which tends to 0 as $\delta \to 0$ by (4.4.18). We bound II and III as [DP16]

$$II + III \lesssim r^{-1}\int_{B_{5r/4}\cap E_\delta} |u|^{p-1}|\nabla u| + \delta^{p-1}r^{-1}\int_{B_{5r/4}\setminus E_\delta} |\nabla u|.$$

By using Cauchy-Schwarz the last term tends to 0 as $\delta \to 0$ by (4.4.18) and so we can incorporate it into the $o(1)$ term. To control the other term we use Hölder's inequality and the ε-Cauchy inequality to show

$$r^{-1}\int_{B_{5r/4}\cap E_\delta}|u|^{p-1}|\nabla u| \le r^{-1}\left(\int_{B_{5r/4}}|u|^p\right)^{(p-1)/p}\left(\int_{B_{5r/4}}|\nabla u|^p\right)^{1/p}$$
$$\le C_\varepsilon r^{-2}\int_{B_{5r/4}}|u|^p + \varepsilon r^{p-2}\int_{B_{5r/4}}|\nabla u|^p.$$

Therefore

$$II + III \le C_\varepsilon r^{-2}\int_{B_{5r/4}}|u|^p + \varepsilon r^{p-2}\int_{B_{5r/4}}|\nabla u|^p + o(1).$$

Now we turn to I and use the same idea as the proof of (4.4.18) in [Lan99, (3.3)] to show I converges as expected. By splitting the integral with the set E_δ, using $\delta^{p-2} \le |u|^{p-2}$ on $B_{\alpha r}\setminus E_\delta$ (since $p<2$), and the smoothness of u (which implies $|u|^{p-2}uu_t \in L^1(B_{\alpha r})$) we obtain

$$\int_{B_{\alpha r}}\rho_\delta^2\left(|u|\right)uu_t = \int_{B_{\alpha r}\cap E_\delta}|u|^{p-2}uu_t + \delta^{p-2}\int_{B_{\alpha r}\setminus E_\delta}uu_t$$
$$\le \int_{B_{\alpha r}\cap E_\delta}|u|^{p-2}uu_t + \int_{B_{\alpha r}\setminus E_\delta}|u|^{p-2}uu_t \le \int_{B_{\alpha r}}|u|^{p-1}|u_t| < \infty.$$

Therefore by the dominated convergence theorem

$$\int_{B_{\alpha r}}\rho_\delta^2\left(|u|\right)uu_t \to \int_{B_{\alpha r}}|u|^{p-2}uu_t. \tag{4.4.19}$$

We change from working with balls to integrating over parabolic cubes $Q_{\alpha r}\subset\mathbb{R}^n\times\mathbb{R}$ and denote by $Q_{\alpha r}|_s$ the cube $Q_{\alpha r}$ restrict to the hypersurface $\{t=s\}$. Using the fundamental theorem of calculus we obtain in the limit that

$$\begin{aligned}\int_{B_{\alpha r}}|u|^{p-2}uu_t &\sim \int_{B_{\alpha r}}\frac{\partial}{\partial t}\left(|u|^p\right)\,\mathrm{d}t\,\mathrm{d}X\\ &\le \int_{Q_{\alpha r}}\frac{\partial}{\partial t}\left(|u|^p\right)\,\mathrm{d}t\,\mathrm{d}X = \int_{t_0-(\alpha r)^2}^{t_0+(\alpha r)^2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{Q_{\alpha r}|_s}|u|^p\,\mathrm{d}X\,\mathrm{d}s\\ &\le \|u\|^p_{L^p_X\left(Q_{\alpha r}|_{t_0+(\alpha r)^2}\right)} + \|u\|^p_{L^p_X\left(Q_{\alpha r}|_{t_0-(\alpha r)^2}\right)}.\end{aligned} \tag{4.4.20}$$

Observe that (4.4.20) holds for all time restricted cubes $Q_{\alpha r}|_{t_0\pm(\alpha r)^2}$ with $\alpha\in[1,1.1]$. Once again we average over these cubes to show

$$\int_{B_{\alpha r}}|u|^{p-2}uu_t \lesssim \frac{1}{r^2}\int_{Q_{1.1\alpha r}}|u|^p\,\mathrm{d}X\,\mathrm{d}t.$$

Since $Q_{1.1\alpha r}\subset B_{2r}$, in the limit as $\delta\to0$

$$I \lesssim \frac{1}{r^2}\int_{B_{2r}}|u|^p\,\mathrm{d}X\,\mathrm{d}t.$$

Therefore grouping the estimates together we have the following bound

$$\lambda'\int_{B_r\cap E_\delta}|u|^{p-2}|\nabla u|^2 \lesssim C_\varepsilon r^{-2}\int_{B_{2r}}|u|^p + \varepsilon r^{p-2}\int_{B_{5r/4}}|\nabla u|^p + o(1). \tag{4.4.21}$$

We let $\delta\to0$ to obtain

$$r^2\int_{B_r\cap\{u\neq0\}}|u|^{p-2}|\nabla u|^2 \lesssim C_\varepsilon\int_{B_{2r}}|u|^p + \varepsilon r^p\int_{B_{5r/4}}|\nabla u|^p, \tag{4.4.22}$$

where the implicit constant depends only on λ,Λ and K and not on the smoothness of A or B. Recalling the convention that $|u|^{p-2}|\nabla u|^2=0$ when $\nabla u=0$ the integral on the left hand side of (4.4.22) can be taken over the set $B_r \setminus \{u = 0, \nabla u \neq 0\}$. However this set has the same measure as B_r so after taking averages we conclude

$$r^2 \fint_{B_r} |u|^{p-2}|\nabla u|^2 \lesssim C_\varepsilon \fint_{B_{2r}} |u|^p + \varepsilon r^p \fint_{B_{5r/4}} |\nabla u|^p. \tag{4.4.23}$$

For the final step in proving (4.4.12) is to use Hölder's inequality and the Caccioppoli inequality (lemma 2.2.1) on the last term of (4.4.23)

$$r^p \fint_{B_{5r/4}} |\nabla u|^p \lesssim \left(r^2 \fint_{B_{5r/4}} |\nabla u|^2 \right)^{p/2} \lesssim \left(\fint_{B_{2r}} |u|^2 \right)^{p/2}. \tag{4.4.24}$$

We conclude the proof of (4.4.12) by combining (4.4.23) and (4.4.24), and $|u|^{(p-2)/2}u \in W^{1,2}_{\mathrm{loc}}$ by the Caccioppoli inequality.

To show (4.4.13) we use a bootstrapping argument and the Sobolev embedding theorem. First assume that $n > 2$, let $p^* = \frac{np}{n-2}$ and let $2^* = \frac{n2}{n-2}$ then by the Sobolev embedding theorem, lemma 4.4.7 and (4.4.12)

$$\begin{aligned} \left(\fint_{B_r} |u|^{p^*} \right)^{1/2^*} &\lesssim r \left(\fint_{B_r} \left| \nabla \left(|u|^{p/2-1} u \right) \right|^2 \right)^{1/2} + \left(\fint_{B_r} |u|^p \right)^{1/2} \\ &\lesssim r \left(\fint_{B_r} |u|^{p-2} |\nabla u|^2 \right)^{1/2} + \left(\fint_{B_{2r}} |u|^p \right)^{1/2} \\ &\lesssim C_\varepsilon \left(\fint_{B_{2r}} |u|^p \right)^{1/2} + \varepsilon \left(\fint_{B_{2r}} |u|^2 \right)^{1/2}. \end{aligned} \tag{4.4.25}$$

Therefore raising both sides to $2/p$ and using that $\frac{2}{2^*} = \frac{p}{p^*}$

$$\left(\fint_{B_r} |u|^{p^*} \right)^{1/p^*} \lesssim C_\varepsilon \left(\fint_{B_{2r}} |u|^p \right)^{1/p} + \varepsilon \left(\fint_{B_{2r}} |u|^2 \right)^{1/2}. \tag{4.4.26}$$

We proved this for $B_{\alpha r}$ when $\alpha = 2$ but by adjusting the integrals we averaged over the same result holds for $B_{\alpha r}$ for any $\alpha > 1$, with the new constant implicitly depending on α. So by iterating this k times for $p_k = 2\left(\frac{n}{n-2}\right)^k$, $p_k \leq p < p_{k-1} < \cdots < p_0 = 2$, and using Hölder's inequality we obtain

$$\begin{aligned} \left(\fint_{B_r} |u|^2 \right)^{1/2} &\lesssim C_\varepsilon \left(\fint_{B_{\alpha r}} |u|^{p_1} \right)^{1/p_1} + \varepsilon \left(\fint_{B_{2r}} |u|^2 \right)^{1/2} \\ &\lesssim \cdots \lesssim C_\varepsilon^k \left(\fint_{B_{\alpha^k r}} |u|^{p_k} \right)^{1/p_k} + \varepsilon^k \left(\fint_{B_{2r}} |u|^2 \right)^{1/2} \\ &\lesssim C_\varepsilon^k \left(\fint_{B_{\alpha^k r}} |u|^p \right)^{1/p} + \varepsilon^k \left(\fint_{B_{2r}} |u|^2 \right)^{1/2}. \end{aligned}$$

We can choose ε as small as we want and independent of the finite number of iteration steps. If we fix α such that $\alpha^k = 2$ then this proves (4.4.13) for $n > 2$.

For the case $n = 1, 2$ the Sobolev embedding theorem allows us to replace p^* in (4.4.26) by any $1 < q < \infty$. This gives the result directly with the implicit constant independent of p.

Finally, since no constants depend on the smoothness of A or B, we can remove the smoothness assumption by the same argument as in [HL01]. We suppose A is just elliptic and bounded, and B satisfies (1.0.3) then we approximate A and B by smooth matrices and vectors respectively. For each smooth approximation we have (4.4.12) and (4.4.13) and then passing to the limit we obtain analogous estimates for $W^{1,2}_{\mathrm{loc}}$ solutions u of $Lu = 0$ with the constants having the same dependence as before. □

It follows that the p-adapted square function $S_{p,a}$ is well defined if we truncate the integral away from the boundary. Dindoš and Hwang [DH18] also considered an area function and established in [DH18, Lemma 5.2] that this area function can be controlled by the usual square function. The case $1 < p < 2$ is significantly more complicated so for this reason we focus only on non-negative solutions u.

We fix a boundary point $(Y, s) \in \partial\Omega$ and consider $A_{p,a}(Y, s)$. Clearly, the non-tangential cone $\Gamma_a(Y, s)$ can be covered by non-overlapping collection of Whitney cubes $\{Q_i\}$ with the following properties:

$$\Gamma_a(Y, s) \subset \bigcup_i Q_i \subset \Gamma_{2a}(Y, s), \quad r_i := \operatorname{diam}(Q_i) \sim \operatorname{dist}(Q_i, \partial\Omega), \quad 4Q_i \subset \Omega, \tag{4.4.27}$$

and the cubes $\{2Q_i\}$ have bounded overlap. It follows that

$$\begin{aligned} [A_{p,a}(Y, s)]^p &\lesssim \sum_i (r_i)^{2-n} \int_{Q_i} |u_t|^2 u^{p-2} \, \mathrm{d}X \, \mathrm{d}t \\ &\lesssim \sum_i (r_i)^{2-n} \int_{Q_i} |\nabla^2 u|^2 u^{p-2} + \left(|\nabla A|^2 + |B|^2\right) |\nabla u|^2 u^{p-2} \, \mathrm{d}X \, \mathrm{d}t. \end{aligned} \tag{4.4.28}$$

We need the following estimate on each Q_i.

Proposition 4.4.9 (p-adapted Caccioppoli inequality for the second gradient)**.** *Assume the ellipticity condition* (1.0.2) *and that the coefficients A and B of* (4.1.1) *satisfy the conditions*

$$|\nabla A(X, t)| \le K/\delta(X, t) \quad \textit{and} \quad |B(X, t)| \le K/\delta(X, t),$$

for some uniform constant $K > 0$. For all non-negative solutions u of (4.1.1) *and any parabolic cube Q such that $4Q \subset \Omega$ we have the following estimate for some constant $\varepsilon = \varepsilon(p, \lambda, n) > 0$, with $\varepsilon \to 0$ as $p \to 1$,*

$$\begin{aligned} &\sup_\tau \int_{Q\cap\{t=\tau\}} |\nabla u|^2 u^{p-2} \, \mathrm{d}X + \varepsilon \int_Q |\nabla^2 u|^2 u^{p-2} \, \mathrm{d}X \, \mathrm{d}t + \varepsilon \int_Q |\nabla u|^4 u^{p-4} \, \mathrm{d}X \, \mathrm{d}t \\ &\quad\lesssim r^{-2} \int_{2Q} |\nabla u|^2 u^{p-2} \, \mathrm{d}X \, \mathrm{d}t, \end{aligned} \tag{4.4.29}$$

where $r = \operatorname{diam}(Q)$.

Proof. Since we assume differentiability of the matrix A in the spatial variables we may also assume that A is symmetric. As before, we first assume that the coefficients A and B are smooth and hence u is smooth. Later we remove this smoothness assumption by an approximation argument as the implicit constant only depends on the ellipticity constants, K and the dimension n.

Let us denote by $W = (w_k)$, where $w_k = \partial_k u$ for $k = 0, 1, \dots, n-1$. Differentiating (4.1.1) we obtain the following PDE for each w_k

$$(w_k)_t - \operatorname{div}(A\nabla w_k) = \operatorname{div}((\partial_k A)W) + \partial_k(B \cdot W). \tag{4.4.30}$$

We multiply (4.4.30) by $w_k u^{p-2}\zeta^2$, integrate over $2Q$ and then integrate by parts. Here $0 \le \zeta \le 1$ is a smooth cut-off function equal to 1 on Q, vanishing outside $2Q$ and satisfying $r|\nabla\zeta| + r^2|\zeta_t| \le C$ for some $C > 0$ independent of Q. This gives

$$\begin{aligned} &\int_{2Q} (w_k)_t w_k u^{p-2} \zeta^2 \, \mathrm{d}X \, \mathrm{d}t + \int_{2Q} a_{ij} (\partial_j w_k) \partial_i (w_k u^{p-2} \zeta^2) \, \mathrm{d}X \, \mathrm{d}t \\ &= -\int_{2Q} (\partial_k a_{ij}) w_j \partial_i (w_k u^{p-2}\zeta^2) \, \mathrm{d}X \, \mathrm{d}t - \int_{2Q} b_i w_i \partial_k (w_k u^{p-2}\zeta^2) \, \mathrm{d}X \, \mathrm{d}t. \end{aligned} \tag{4.4.31}$$

We rearrange and group similar terms together

$$
\begin{aligned}
&\frac{1}{2}\int_{2Q}\left[(w_k u^{p/2-1}\zeta)^2\right]_t \,\mathrm{d}X\,\mathrm{d}t + \int_{2Q} A\left(\nabla(w_k\zeta)u^{p/2-1}\right)\cdot\left(\nabla(w_k\zeta)u^{p/2-1}\right)\,\mathrm{d}X\,\mathrm{d}t\\
&\quad + (p-2)\int_{2Q} A\left(\nabla(w_k\zeta)u^{p/2-1}\right)\cdot\left((\nabla u)w_k u^{p/2-2}\zeta\right)\,\mathrm{d}X\,\mathrm{d}t - \frac{p-2}{2}\int_{2Q} w_k^2 u^{p-3}u_t\zeta^2\,\mathrm{d}X\,\mathrm{d}t\\
&= \int_{2Q}|w_k|^2 u^{p-2}\zeta\zeta_t\,\mathrm{d}X\,\mathrm{d}t + \int_{2Q}|w_k|^2u^{p-2}A\nabla\zeta\cdot\nabla\zeta\,\mathrm{d}X\,\mathrm{d}t - \int_{2Q} b_i w_i\partial_k(w_k\zeta)u^{p-2}\zeta\,\mathrm{d}X\,\mathrm{d}t\\
&\quad - (p-2)\int_{2Q} b_i w_i\left((\partial_k u)w_k u^{p/2-2}\zeta\right)u^{p/2-1}\zeta\,\mathrm{d}X\,\mathrm{d}t - \int_{2Q} b_i w_i w_k u^{p-2}\zeta\zeta_k\,\mathrm{d}X\,\mathrm{d}t\\
&\quad - \int_{2Q}(\partial_k a_{ij})w_j w_k u^{p-2}\zeta\zeta_i\,\mathrm{d}X\,\mathrm{d}t - \int_{2Q}(\partial_k a_{ij})w_j(\partial_i w_k\zeta)u^{p-2}\zeta\,\mathrm{d}X\,\mathrm{d}t\\
&\quad - (p-2)\int_{2Q}(\partial_k a_{ij})w_j\left((\partial_i u)w_k u^{p/2-2}\zeta\right)u^{p/2-1}\zeta\,\mathrm{d}X\,\mathrm{d}t.
\end{aligned}\tag{4.4.32}
$$

All the terms after the equal sign are error terms since they either contain a derivative of ζ, or the coefficients ∇A or B. These will be handled using the Cauchy-Schwarz inequality and the estimates for $|\nabla A|, |B| \le K/r$. The four main terms are on the left hand side of (4.4.32). The term that needs further work is the fourth term and we use the PDE (4.1.1) for u_t. This gives

$$
\begin{aligned}
-\frac{p-2}{2}\int_{2Q} w_k^2u^{p-3}u_t\zeta^2\,\mathrm{d}X\,\mathrm{d}t = &-\frac{p-2}{2}\int_{2Q} w_k^2u^{p-3}\operatorname{div}(A\nabla u)\zeta^2\,\mathrm{d}X\,\mathrm{d}t\\
&- \frac{p-2}{2}\int_{2Q} w_k^2u^{p-3}B\cdot W\zeta^2\,\mathrm{d}X\,\mathrm{d}t.
\end{aligned}\tag{4.4.33}
$$

Again the second term of (4.4.33) is an error term. For the first term of (4.4.33) we observe the equality

$$
u^{p-3}\operatorname{div}(A\nabla u) = \operatorname{div}\left(A(\nabla u)u^{p-3}\right) - (p-3)A\left((\nabla u)u^{p/2-2}\right)\cdot\left((\nabla u)u^{p/2-2}\right).
$$

It follows (by integrating by parts)

$$
\begin{aligned}
&-\frac{p-2}{2}\int_{2Q} w_k^2u^{p-3}\operatorname{div}(A\nabla u)\zeta^2\,\mathrm{d}X\,\mathrm{d}t\\
&= (p-2)\int_{2Q} A\left(\nabla(w_k\zeta)u^{p/2-1}\right)\cdot\left((\nabla u)w_k u^{p/2-2}\zeta\right)\,\mathrm{d}X\,\mathrm{d}t\\
&\quad + \frac{(2-p)(3-p)}{2}\int_{2Q} A\left((\nabla u)w_k u^{p/2-2}\zeta\right)\cdot\left((\nabla u)w_k u^{p/2-2}\zeta\right)\,\mathrm{d}X\,\mathrm{d}t.
\end{aligned}\tag{4.4.34}
$$

We now group all main terms together; these are the first, second and third terms on the left-hand side of (4.4.32) and the terms on the right hand side of (4.4.34). This gives

$$
\begin{aligned}
&\text{LHS of (4.4.32)} = \frac{1}{2}\int_{2Q}\left[\left(w_k u^{p/2-1}\zeta\right)^2\right]_t \mathrm{d}X\,\mathrm{d}t\\
&\quad+\int_{2Q} A\left(\nabla(w_k\zeta)u^{p/2-1}\right)\cdot\left(\nabla(w_k\zeta)u^{p/2-1}\right)\mathrm{d}X\,\mathrm{d}t\\
&\quad+2(p-2)\int_{2Q} A\left(\nabla(w_k\zeta)u^{p/2-1}\right)\cdot\left((\nabla u)w_k u^{p/2-2}\zeta\right)\mathrm{d}X\,\mathrm{d}t\\
&\quad+\frac{(2-p)(3-p)}{2}\int_{2Q} A\left((\nabla u)w_k u^{p/2-2}\zeta\right)\cdot\left((\nabla u)w_k u^{p/2-2}\zeta\right)\mathrm{d}X\,\mathrm{d}t\\
&=\frac{1}{2}\int_{2Q}\left[\left(w_k u^{p/2-1}\zeta\right)^2\right]_t \mathrm{d}X\,\mathrm{d}t\\
&\quad+\left(1-\frac{2(2-p)}{3-p}\right)\int_{2Q} A\left(\nabla(w_k\zeta)u^{p/2-1}\right)\cdot\left(\nabla(w_k\zeta)u^{p/2-1}\right)\mathrm{d}X\,\mathrm{d}t\\
&\quad+\int_{2Q} A\left(\sqrt{\tfrac{2(2-p)}{3-p}}\left[\nabla(w_k\zeta)u^{p/2-1}\right]-\sqrt{\tfrac{(2-p)(3-p)}{2}}\left[(\nabla u)w_k u^{p/2-2}\zeta\right]\right)\\
&\qquad\qquad\times\left(\sqrt{\tfrac{2(2-p)}{3-p}}\left[\nabla(w_k\zeta)u^{p/2-1}\right]-\sqrt{\tfrac{(2-p)(3-p)}{2}}\left[(\nabla u)w_k u^{p/2-2}\zeta\right]\right)\mathrm{d}X\,\mathrm{d}t\\
&\geq\frac{1}{2}\int_{2Q}\left[(w_k u^{p/2-1}\zeta)^2\right]_t \mathrm{d}X\,\mathrm{d}t+\frac{(p-1)\lambda}{3-p}\int_{2Q}\left|\nabla(w_k\zeta)u^{p/2-1}\right|^2\mathrm{d}X\,\mathrm{d}t.
\end{aligned}
\tag{4.4.35}
$$

Here we have first completed the square (using symmetry of A), and then used the ellipticity of the matrix A. The important point is that for all $1<p<2$ the coefficient $\frac{(p-1)\lambda}{3-p}$ is positive.

We also note that we could have completed the square differently and obtained instead of (4.4.35) the estimate

$$
\begin{aligned}
\text{LHS of (4.4.32)} \geq{}& \frac{1}{2}\int_{2Q}\left[(w_k u^{p/2-1}\zeta)^2\right]_t \mathrm{d}X\,\mathrm{d}t\\
&+\frac{(p-1)(2-p)\lambda}{2}\int_{2Q}\left|(\nabla u)w_k u^{p/2-2}\zeta\right|^2\mathrm{d}X\,\mathrm{d}t.
\end{aligned}
\tag{4.4.36}
$$

It follows that we could average (4.4.35) and (4.4.36) and have both

$$
\int_{2Q}\left|\nabla(w_k\zeta)u^{p/2-1}\right|^2\mathrm{d}X\,\mathrm{d}t \quad\text{and}\quad \int_{2Q}\left|(\nabla u)w_k u^{p/2-2}\zeta\right|^2\mathrm{d}X\,\mathrm{d}t
$$

in the estimate with small positive constants.

Now we briefly mention how all the error terms of (4.4.32), (4.4.33) and (4.4.35) can be handled. Some can be immediately estimated from above by

$$
r^{-2}\int_{2Q}|W|^2u^{p-2}\,\mathrm{d}X\,\mathrm{d}t,
$$

where the scaling factor r^{-2} comes from the estimates on $\nabla\zeta$, ζ_t, $|\nabla A|$ and $|B|$. For other terms (for example the third term on the third line of (4.4.32) or the first term on the fourth line) we use Cauchy-Schwarz. One of the terms in the product is

$$
\left(r^{-2}\int_{2Q}|W|^2u^{p-2}\,\mathrm{d}X\,\mathrm{d}t\right)^{1/2},
$$

while the other term is one of

$$
\left(\int_{2Q}\left|\nabla(w_k\zeta)u^{p/2-1}\right|^2\mathrm{d}X\,\mathrm{d}t\right)^{1/2} \quad\text{or}\quad \left(\int_{2Q}\left|(\nabla u)w_k u^{p/2-2}\zeta\right|^2\mathrm{d}X\,\mathrm{d}t\right)^{1/2}.
$$

It follows using the ε-Cauchy-Schwarz inequality that we can hide these on the left-hand side of (4.4.32). Finally, we put everything together by summing over all k and recalling that $W = \nabla u$ to obtain (4.4.29) for smooth coefficients.

The calculations above clearly work for solutions u with uniform bound $u \geq \delta > 0$. Hence considering $v_\delta = u + \delta$ and then taking the limit $\delta \to 0+$ using Fatou's lemma yields (4.4.29) for all non-negative u, where we have used the convention that $|\nabla u|^2 u^{p-2} = 0$ whenever $u = 0$ and $\nabla u = 0$ with a similar convention for the second gradient.

As before we remove the smoothness assumption by a small modification to the standard approximation argument, which can be found in [HL01]. We first smoothly approximate the coefficients A and B by A_j and B_j to give us smooth solutions u_j. Therefore the following quantities are uniformly bounded

$$\int_Q |\nabla^2 u_j| \leq C_1 \qquad \text{and} \qquad \int_Q |\nabla u_j|^4 \leq C_2. \tag{4.4.37}$$

By extracting weakly convergent subsequences, using Fatou's lemma, the uniform convergence of $u_j \to u$ on compact subsets and the maximum principle we have that (4.4.37) converges as desired and (4.4.29) holds. □

After using (4.4.29) in (4.4.28) we can conclude the following.

Lemma 4.4.10. *Let u be a non-negative solution of* (4.1.1) *with the matrix A satisfying the ellipticity hypothesis and the coefficients satisfying the bound $|\nabla A|, |B| \leq K/\delta$. Given $a > 0$ there exists a constant $C = (\Lambda, \lambda, a, K, p, n)$ such that*

$$A_{p,a}(u)(X,t) \leq C S_{p,2a}(u)(X,t). \tag{4.4.38}$$

From this we have the global estimate

$$\|A_{p,a}(u)\|^p_{L^p(\partial\Omega)} \leq C_2 \|S_{p,a}(u)\|^p_{L^p(\partial\Omega)}. \tag{4.4.39}$$

As far as the proof goes, we again consider $v_\delta = u + \delta$ and then take the limit using Fatou's lemma.

4.5 Bounding the p-adapted Square Function by the Non-tangential Maximal Function

We slightly abuse notation and only work on a Carleson region $T(Q_r)$ of parabolic boundary balls in the upper half space U even though we formulate the following lemmas on any admissible domain Ω. The equivalence of these formulations via the pullback map ρ is discussed in section 4.2.4 and [DH18], and hence we omit the details.

In addition, for ease of notation and not to confuse the following proofs, we assume that the p-adapted square function is well defined and integrable for solutions. This allows us to perform the following proof and integrate by parts. At the moment we have only showed it is locally integrable in the interior by theorem 4.4.3. However, we formally justify the steps taken in the proofs of lemmas 4.5.1 and 4.5.2 below in remark 4.5.3. Hence this shows the finiteness of $S_{p,a}(u)$ and $\|S_{p,a}(u)\|_{L^p}$ by Tonelli's theorem and lemma 4.5.1 below.

We start with a local bound of the p-adapted square function by the non-tangential maximal function.

Lemma 4.5.1. *Let Ω be an admissible domain from definition 4.2.20 with character (ℓ, η, N, d). Let $1 < p < 2$ and u be a non-negative solution of* (4.1.1)*, with the Carleson conditions* (4.1.7) *and* (4.1.8) *on the coefficients A and B and the constant r_0 from theorem 4.1.6. Then there exists a constant $C = C(\lambda, \Lambda, N, d)$ such that for any solution u with boundary data f on any cube $Q_r \subset \partial\Omega$ with $r \leq \min\{d/4, r_0/4\}$ we have*

$$\int_{T(Q_r)} |\nabla u|^2 |u|^{p-2} x_0 \,\mathrm{d}x_0 \,\mathrm{d}x \,\mathrm{d}t \leq C(1 + \|\mu\|_{C,r_0})(1 + \ell^2) \int_{Q_{2r}} \left(N^{2r}\right)^2 (u) \,\mathrm{d}x \,\mathrm{d}t. \tag{4.5.1}$$

In addition we have the following global result.

Lemma 4.5.2. *Let Ω be an admissible domain with smooth boundary $\partial\Omega$. Let $1 < p < 2$ and u be a non-negative solution of* (4.1.1) *satisfying* (4.2.63), (4.2.64), (4.2.66) *and* (4.2.67) *with Dirichlet boundary data $f \in L^p(\partial\Omega)$. Then there exists positive constants C_1 and C_2 independent of u such that for small $r_0 > 0$ we have*

$$\begin{aligned}
&\frac{C_1}{2}\int_0^{r_0/2}\int_{\partial\Omega}|\nabla u|^2|u|^{p-2}x_0\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0 + \frac{2}{r_0}\int_0^{r_0}\int_{\partial\Omega}u^p(x_0,x,t)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0\\
&\leq \int_{\partial\Omega}u^p(r_0,x,t)\,\mathrm{d}x\,\mathrm{d}t + \int_{\partial\Omega}u^p(0,x,t)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + C_2\left(\|\mu_1\|_{C,2r} + \|\mu_2\|_{C,2r} + \|\mu_2\|_{C,2r}^{1/2}\right)\int_{\partial\Omega}\left(N^{2r}(u)\right)^p\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{4.5.2}$$

Proof of lemmas 4.5.1 and 4.5.2. Let $Q_r(y,s)$ be a parabolic cube on the boundary with $r < d$ and let ζ be a smooth cut off function independent of the x_0 variable. As long as there is no ambiguity we suppress the argument of Q_r and extensively use the Einstein summation convention. Let ζ be supported in Q_{2r}, equal 1 in Q_r and satisfy the estimate $r|\nabla\zeta|+r^2|\partial_t\zeta| \leq C$ for some constant C.

We start by estimating

$$\int_0^r\int_{Q_{2r}}|u|^{p-2}\frac{a_{ij}}{a_{00}}(\partial_i u)(\partial_j u)\zeta^2 x_0\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0, \tag{4.5.3}$$

where by ellipticity we have

$$\frac{\lambda}{\Lambda}\int_0^r\int_{Q_r}|\nabla u|^2|u|^{p-2}x_0\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0 \leq \int_0^r\int_{Q_{2r}}|u|^{p-2}\frac{a_{ij}}{a_{00}}(\partial_i u)(\partial_j u)\zeta^2 x_0\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0.$$

Now we integrate by parts whilst noting that $\nu = (1,0,0,\dots,0)$ since the domain is $\{x_0 > 0\}$

$$\begin{aligned}
&\int_0^r\int_{Q_{2r}}|u|^{p-2}\frac{a_{ij}}{a_{00}}(\partial_i u)(\partial_j u)\zeta^2 x_0\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0\\
&= \frac{1}{p}\int_{Q_{2r}}\frac{a_{0j}}{a_{00}}\partial_j\left(|u(r,x,t)|^p\right)r\zeta^2\,\mathrm{d}x\,\mathrm{d}t - \int_0^r\int_{Q_{2r}}\frac{1}{a_{00}}|u|^{p-2}u\partial_i(a_{ij}\partial_j u)\zeta^2 x_0\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0\\
&- \int_0^r\int_{Q_{2r}}\partial_i\left(\frac{1}{a_{00}}\right)|u|^{p-2}ua_{ij}\partial_j u\zeta^2 x_0\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0 - 2\int_0^r\int_{Q_{2r}}\frac{a_{ij}}{a_{00}}|u|^{p-2}u(\partial_j u)\zeta\partial_i\zeta x_0\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0\\
&- \int_0^r\int_{Q_{2r}}\frac{a_{0j}}{a_{00}}|u|^{p-2}u(\partial_j u)\zeta^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0 - \int_0^r\int_{Q_{2r}}\frac{a_{ij}}{a_{00}}\partial_i\left(|u|^{p-2}\right)u(\partial_j u)\zeta^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0\\
&= I + II + III + IV + V + VI.
\end{aligned} \tag{4.5.4}$$

Our strategy is to further estimate all these terms and then group similar terms together. First consider II, we use that u is a solution to (4.1.1)

$$\begin{aligned}
II &= -\int_0^r\int_{Q_{2r}}\frac{1}{a_{00}}|u|^{p-2}uu_t\zeta^2 x_0\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0 + \int_0^r\int_{Q_{2r}}\frac{1}{a_{00}}|u|^{p-2}ub_i\partial_i u\zeta^2 x_0\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0\\
&= II_1 + II_2.
\end{aligned}$$

Using the identity $2x_0 = \partial_0 x_0^2$ we integrate by parts in x_0 to obtain

$$\begin{aligned}
II_1 &= -\frac{1}{2}\int_0^r \int_{Q_{2r}} \frac{1}{a_{00}} |u|^{p-2} u u_t \zeta^2 \partial_0 x_0^2 \,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0 \\
&= -\frac{1}{2}\int_{Q_{2r}} \frac{1}{a_{00}} |u(r,x,t)|^{p-2} u(r,x,t) u_t(r,x,t) \zeta^2 r^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \frac{1}{2}\int_0^r \int_{Q_{2r}} \partial_0 \left(\frac{1}{a_{00}}\right) |u|^{p-2} u u_t \zeta^2 x_0^2 \,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0 \\
&\quad + \frac{p-1}{2}\int_0^r \int_{Q_{2r}} \frac{1}{a_{00}} |u|^{p-2} \partial_0 u u_t \zeta^2 x_0^2 \,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0 + \frac{1}{2}\int_0^r \int_{Q_{2r}} \frac{1}{a_{00}} |u|^{p-2} u \partial_0 \partial_t u \zeta^2 x_0^2 \,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0 \\
&= II_{11} + II_{12} + II_{13} + II_{14}.
\end{aligned}$$

Consider the boundary term II_{11}. If we integrate by parts in t then

$$\begin{aligned}
II_{11} &= -\frac{1}{4}\int_{Q_{2r}} \frac{1}{a_{00}} |u(r,x,t)|^{p-2} \partial_t \left(u^2(r,x,t)\right) \zeta^2 r^2 \,\mathrm{d}x\,\mathrm{d}t \\
&= \frac{1}{4}\int_{Q_{2r}} \partial_t \left(\frac{1}{a_{00}}\right) |u(r,x,t)|^p \zeta^2 r^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_{Q_{2r}} \frac{1}{a_{00}} |u(r,x,t)|^p \zeta \zeta_t r^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \frac{p-2}{4}\int_{Q_{2r}} \frac{1}{a_{00}} |u(r,x,t)|^{p-2} u(r,x,t) u_t(r,x,t) \zeta^2 r^2 \,\mathrm{d}x\,\mathrm{d}t \\
&= II_{111} + II_{112} + II_{113}.
\end{aligned}$$

Since $p < 2$, so $p-2 < 0$, we can absorb II_{113} into II_{11} and save II_{12} to bound later on. Considering II_{14}, we swap the order of differentiation on $\partial_0 \partial_t u$ and integrate by parts in t to show

$$\begin{aligned}
II_{14} &= \frac{1}{2}\int_0^r \int_{Q_{2r}} \frac{1}{a_{00}} |u|^{p-2} u \partial_t \partial_0 u \zeta^2 x_0^2 \,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0 \\
&= -\frac{1}{2}\int_0^r \int_{Q_{2r}} \partial_t \left(\frac{1}{a_{00}}\right) |u|^{p-2} u \partial_0 u \zeta^2 x_0^2 \,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0 \\
&\quad - \frac{p-1}{2}\int_0^r \int_{Q_{2r}} \frac{1}{a_{00}} |u|^{p-2} u_t \partial_0 u \zeta^2 x_0^2 \,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0 - \int_0^r \int_{Q_{2r}} \frac{1}{a_{00}} |u|^{p-2} u \partial_0 u \zeta \zeta_t x_0^2 \,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0 \\
&= II_{141} + II_{142} + II_{143}.
\end{aligned}$$

Observe that $II_{142} = -II_{13}$ so these terms cancel. We bound II_{141} by

$$\begin{aligned}
II_{141} &= \frac{1}{2}\int_0^r \int_{Q_{2r}} \frac{\partial_t a_{00}}{a_{00}^2} |u|^{p-2} u \partial_0 u \zeta^2 x_0^2 \,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0 \\
&\lesssim \left(\int_0^r \int_{Q_{2r}} |A_t|^2 |u|^p x_0^3 \zeta^2 \,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0\right)^{1/2} \left(\int_0^r \int_{Q_{2r}} |\nabla u|^2 |u|^{p-2} x_0 \zeta^2 \,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0\right)^{1/2}.
\end{aligned}$$

The two parts of II_1 we have left to bound are II_{112} and II_{143}. Both of these integrals involve $\zeta\zeta_t$ and therefore if ζ is a partition of unity when we sum over that partition these terms sum to 0.

The terms II_2 and III are simply dealt with by

$$\begin{aligned}
II_2 &\lesssim \left(\int_0^r \int_{Q_{2r}} |B|^2 |u|^p x_0 \zeta^2 \,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0\right)^{1/2} \left(\int_0^r \int_{Q_{2r}} |\nabla u|^2 |u|^{p-2} x_0 \zeta^2 \,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0\right)^{1/2}, \\
III &\lesssim \left(\int_0^r \int_{Q_{2r}} |\nabla A|^2 |u|^p x_0 \zeta^2 \,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0\right)^{1/2} \left(\int_0^r \int_{Q_{2r}} |\nabla u|^2 |u|^{p-2} x_0 \zeta^2 \,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0\right)^{1/2}.
\end{aligned}$$

The integral in the term IV contains the terms $\zeta \partial_i \zeta$ and as before if ζ is a partition of unity then after summing this term cancels out. Therefore the terms that we have yet to estimate are

I, V and VI.

We consider V in the two cases $j=0$ and $j\neq 0$ separately. Since ζ is independent of x_0 by the fundamental theorem of calculus

$$\begin{aligned} V_{\{j=0\}} &= -\int_0^r \int_{Q_{2r}} |u|^{p-2}u(\partial_0 u)\zeta^2 \,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0 = -\frac{1}{p}\int_0^r \int_{Q_{2r}} \partial_0\left(|u|^p\zeta^2\right)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0 \\ &= \frac{1}{p}\int_{Q_{2r}} |u(0,x,t)|^p\zeta^2\,\mathrm{d}x\,\mathrm{d}t - \frac{1}{p}\int_{Q_{2r}} |u(r,x,t)|^p\zeta^2\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

For the $j\neq 0$ case we use that $\partial_0 x_0 = 1$ and integrate by parts in x_0

$$\begin{aligned} V_{\{j\neq 0\}} &= -\frac{1}{p}\int_0^r \int_{Q_{2r}} \frac{a_{0j}}{a_{00}}\partial_j\left(|u|^p\right)\zeta^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0 = -\frac{1}{p}\int_0^r \int_{Q_{2r}} \frac{a_{0j}}{a_{00}}\partial_j\left(|u|^p\right)\zeta^2\partial_0 x_0\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0 \\ &= -\frac{1}{p}\int_{Q_{2r}} \frac{a_{0j}}{a_{00}}\partial_j\left(|u(r,x,t)|^p\right)\zeta^2 r\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{p}\int_0^r \int_{Q_{2r}} \frac{a_{0j}}{a_{00}}\partial_j\partial_0\left(|u|^p\right)\zeta^2 x_0\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0 \\ &\quad + \frac{1}{p}\int_0^r \int_{Q_{2r}} \partial_0\left(\frac{a_{0j}}{a_{00}}\right)\partial_j\left(|u|^p\right)\zeta^2 x_0\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0 \\ &= V_1 + V_2 + V_3. \end{aligned}$$

The term $V_1 = -I_{\{j\neq 0\}}$ so they cancel out. For V_2 we integrate by parts in x_j

$$\begin{aligned} V_2 &= -\sum_{j\neq 0}\frac{1}{p}\int_{Q_{2r}} \frac{a_{0j}}{a_{00}}\partial_0\left(|u(r,x,t)|^p\right)\zeta^2 r\,\mathrm{d}x\,\mathrm{d}t - \frac{1}{p}\int_0^r \int_{Q_{2r}} \partial_j\left(\frac{a_{0j}}{a_{00}}\right)\partial_0\left(|u|^p\right)\zeta^2 x_0\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0 \\ &\quad - \frac{2}{p}\int_0^r \int_{Q_{2r}} \frac{a_{0j}}{a_{00}}\partial_0\left(|u|^p\right)\zeta\partial_j\zeta x_0\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0 \\ &= V_{21} + V_{22} + V_{23}. \end{aligned}$$

V_{22} and V_3 are of the same type and can be estimated as III by

$$\begin{aligned} &\left|\int_0^r \int_{Q_{2r}} \nabla\left(\frac{a_{0j}}{a_{00}}\right)\nabla\left(|u|^p\right)\zeta^2 x_0\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0\right| \lesssim \int_0^r \int_{Q_{2r}} |u|^{p-1}|\nabla u||\nabla A|\zeta^2 x_0\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0 \\ &\lesssim \left(\int_0^r \int_{Q_{2r}} |\nabla A|^2|u|^p\zeta^2 x_0\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0\right)^{1/2}\left(\int_0^r \int_{Q_{2r}} |\nabla u|^2|u|^{p-2}\zeta^2 x_0\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0\right)^{1/2}. \end{aligned}$$

The final term from (4.5.4) to estimate is VI

$$\begin{aligned} VI &= -\int_0^r \int_{Q_{2r}} \frac{a_{ij}}{a_{00}}\partial_i\left(|u|^{p-2}\right)u(\partial_j u)\zeta^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0 \\ &= (2-p)\int_0^r \int_{Q_{2r}} \frac{a_{ij}}{a_{00}}|u|^{p-2}(\partial_i u)(\partial_j u)\zeta^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0 \end{aligned}$$

and since $2-p<1$ we can hide VI in the left hand side of (4.5.4).

We are now at the stage where we can group together all the similar terms and estimate them. There are 4 different types of terms:

$$\begin{aligned} J_1 &= I_{\{j=0\}} + II_{111} + V_{\{j=0\}} + V_{21} \\ J_2 &= II_{12} \\ J_3 &= II_{141} + II_2 + III + \sum_{j\neq 0} V_{22} + \sum_{j\neq 0} V_3 \\ J_4 &= II_{112} + II_{143} + IV + \sum_{j\neq 0} V_{23}. \end{aligned}$$

The terms containing $|\nabla A|^2$, $|A_t|$ or $|B|$ are Carleson measures and so we use theorem 2.5.11

repeatedly. First we consider J_1, which consists of boundary terms at $(0,x,t)$ and (r,x,t).

$$\begin{aligned}
J_1 &= \frac{1}{p}\int_{Q_{2r}} \partial_0\left(|u(r,x,t)|^p\right)\zeta^2 r\,\mathrm{d}x\,\mathrm{d}t - \frac{1}{4}\int_{Q_{2r}} \frac{\partial_t a_{00}}{a_{00}^2}|u(r,x,t)|^{p-2}u^2(r,x,t)\zeta^2 r^2\,\mathrm{d}x\,\mathrm{d}t \\
&+ \frac{1}{p}\int_{Q_{2r}} |u(0,x,t)|^p\zeta^2\,\mathrm{d}x\,\mathrm{d}t - \frac{1}{p}\int_{Q_{2r}} |u(r,x,t)|^p\zeta^2\,\mathrm{d}x\,\mathrm{d}t \\
&- \sum_{j\neq 0}\frac{1}{p}\int_{Q_{2r}} \frac{a_{0j}}{a_{00}}\partial_0\left(|u(r,x,t)|^p\right)\zeta^2 r\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

The second term in J_1, originating from II_{111}, has the bound

$$\begin{aligned}
II_{111} &= -\frac{1}{4}\int_{Q_{2r}} \frac{\partial_t a_{00}}{a_{00}^2}|u(r,x,t)|^{p-2}u^2(r,x,t)\zeta^2 r^2\,\mathrm{d}x\,\mathrm{d}t \\
&\le \frac{1}{4\lambda^2}\int_{Q_{2r}} |A_t||u(r,x,t)|^p\zeta^2 r^2\,\mathrm{d}x\,\mathrm{d}t \le \frac{\|\mu_2\|_{C,2r}^{1/2}}{\lambda^2}\|N^r(u)\|^p_{L^p(Q_{2r})}.
\end{aligned}$$

We may again use theorem 2.5.11 to bound J_2

$$\begin{aligned}
J_2 &= \frac{1}{2}\int_0^r\int_{Q_{2r}} \partial_0\left(\frac{1}{a_{00}}\right)|u|^{p-2}uu_t\zeta^2x_0^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0 \\
&\le \frac{1}{2\lambda^2}\left(\int_0^r\int_{Q_{2r}} |\nabla A|^2|u|^p x_0\zeta^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0\right)^{1/2}\left(\int_0^r\int_{Q_{2r}} |u_t|^2|u|^{p-2}x_0^3\zeta^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0\right)^{1/2} \\
&\le \frac{1}{\lambda^2}\left(\|\mu_2\|_{C,2r}\|N^r(u)\|^p_{L^p(Q_{2r})}\right)^{1/2}\left(\int_0^r\int_{Q_{2r}} |u_t|^2|u|^{p-2}x_0^3\zeta^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0\right)^{1/2}.
\end{aligned}$$

With a constant $C_3 = C_3(\lambda,\Lambda,n)$ we can bound J_3 by

$$\begin{aligned}
J_3 &\le C_3\left(\int_0^r\int_{Q_{2r}} \left(x_0|\nabla A|^2 + x_0|B|^2 + x_0^3|A_t|^2\right)|u|^p\zeta^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0\right)^{1/2} \\
&\quad\times\left(\int_0^r\int_{Q_{2r}} |\nabla u|^2|u|^{p-2}x_0\zeta^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0\right)^{1/2} \\
&\le C_3\left(\left(\|\mu_1\|_{C,2r}+\|\mu_2\|_{C,2r}\right)\|N^r(u)\|^p_{L^p(Q_{2r})}\right)^{1/2}\left(\int_0^r\int_{Q_{2r}} |\nabla u|^2|u|^{p-2}x_0\zeta^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0\right)^{1/2}.
\end{aligned}$$

Finally, J_4 consists of terms of the type $\zeta\partial_t\zeta$ or $\zeta\partial_i\zeta$. Later we take ζ to be a partition of unity and so when we sum up over the partition all the terms in J_4 sum to 0.

Therefore after all these calculations

$$\begin{aligned}
&\int_0^r\int_{Q_{2r}} |u|^{p-2}\frac{a_{ij}}{a_{00}}(\partial_i u)(\partial_j u)\zeta^2 x_0\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0 = J_1+J_2+J_3+J_4 \\
&\le J_4 + \frac{n\Lambda}{\lambda}\int_{Q_{2r}} \partial_0\left(|u(r,x,t)|^p\right)\zeta^2 r\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_{2r}} |u(0,x,t)|^p\zeta^2\,\mathrm{d}x\,\mathrm{d}t \\
&- \int_{Q_{2r}} |u(r,x,t)|^p\zeta^2\,\mathrm{d}x\,\mathrm{d}t + \frac{\|\mu_2\|_{C,2r}^{1/2}}{\lambda^2}\|N^r(u)\|^p_{L^p(Q_{2r})} \\
&+ \frac{1}{\lambda^2}\left(\|\mu_2\|_{C,2r}\|N^r(u)\|^p_{L^p(Q_{2r})}\right)^{1/2}\left(\int_0^r\int_{Q_{2r}} |u_t|^2|u|^{p-2}x_0^3\zeta^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0\right)^{1/2} \\
&+ C_3\left(\left(\|\mu_1\|_{C,2r}+\|\mu_2\|_{C,2r}\right)\|N^r(u)\|^p_{L^p(Q_{2r})}\right)^{1/2}\left(\int_0^r\int_{Q_{2r}} |\nabla u|^2|u|^{p-2}x_0\zeta^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0\right)^{1/2}.
\end{aligned}\tag{4.5.5}$$

By assuming that Ω is smooth as well as an admissible domain (definition 4.2.20) there exists a collar neighbourhood V of $\partial\Omega$ in $\mathbb{R}^{n+1}$ such that $\Omega\cap V$ can be globally parametrised by

$(0,r)\times\partial\Omega$ for some small $r>0$, see remark 4.2.28 and [DH18] for details. Using definition 4.2.20, there is a collection of charts covering $\partial\Omega$ with bounded overlap, say by M. We consider a partition of unity of these charts ζ_j, with ζ_j having the same definition, support and estimates as ζ before, and $\sum_j \zeta_j = 1$ everywhere. Therefore, when we sum (4.5.5) over this partition of unity the term on the left hand side is bounded below by

$$\frac{1}{\Lambda}\int_0^r \int_{\partial\Omega} |u|^{p-2}(A\nabla u\cdot\nabla u)x_0 \,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0,$$

which is comparable to the truncated p-adapted square function $\|S_p^r(u)\|^p_{L^p(\partial\Omega)}$. Therefore, remembering that after summing $J_4=0$, for any $\varepsilon>0$ we have

$$\begin{aligned}
&\frac{\lambda}{\Lambda}\|S_p^r(u)\|^p_{L^p(\partial\Omega)} \sim \frac{\lambda}{\Lambda}\int_0^r\int_{\partial\Omega}|u|^{p-2}|\nabla u|^2 x_0\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0\\
&\le \frac{n\Lambda}{\lambda}\int_{\partial\Omega}\partial_0\left(|u(r,x,t)|^p\right) r\,\mathrm{d}x\,\mathrm{d}t + \int_{\partial\Omega}|u(0,x,t)|^p\,\mathrm{d}x\,\mathrm{d}t - \int_{\partial\Omega}|u(r,x,t)|^p\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \frac{M\|\mu_2\|^{1/2}_{C,2r}}{\lambda^2}\|N^r(u)\|^p_{L^p(\partial\Omega)} + \frac{\|\mu_2\|_{C,2r}}{4\varepsilon\lambda^2}\|N^r(u)\|^p_{L^p(\partial\Omega)} + \varepsilon\int_0^r\int_{\partial\Omega}|u_t|^2|u|^{p-2}x_0^3\zeta^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0\\
&\quad + C_3\frac{\|\mu_1\|_{C,2r}+\|\mu_2\|_{C,2r}}{4\varepsilon}\|N^r(u)\|^p_{L^p(\partial\Omega)} + \varepsilon\int_0^r\int_{\partial\Omega}|\nabla u|^2|u|^{p-2}x_0\zeta^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0. \qquad (4.5.6)
\end{aligned}$$

By applying lemma 4.4.10 to the p-adapted area function in (4.5.6) we see that the p-adapted square function on the right hand side of (4.5.6) is always multiplied by ε. By choosing ε small enough we can absorb this p-adapted square function into the left hand side yielding

$$\begin{aligned}
C_1\|S_p^r(u)\|^p_{L^p(\partial\Omega)} &\le \int_{\partial\Omega}\partial_0\left(|u(r,x,t)|^p\right) r\,\mathrm{d}x\,\mathrm{d}t + \int_{\partial\Omega}|u(0,x,t)|^p\,\mathrm{d}x\,\mathrm{d}t - \int_{\partial\Omega}|u(r,x,t)|^p\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + C_2\left(\|\mu_1\|_{C,2r}+\|\mu_2\|_{C,2r}+\|\mu_2\|^{1/2}_{C,2r}\right)\|N^r(u)\|^p_{L^p(\partial\Omega)}. \qquad (4.5.7)
\end{aligned}$$

We average (4.5.7) in the r variable over $[0,r_0]$ and use the identity $(\partial_0|u|^p)x_0 = \partial_0(|u|^p x_0) - |u|^p$ to give

$$\begin{aligned}
&C_1\int_0^{r_0}\int_{\partial\Omega}\left(x_0-\frac{x_0^2}{r_0}\right)|\nabla u|^2|u|^{p-2}\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0 + \frac{2}{r_0}\int_0^{r_0}\int_{\partial\Omega}|u(x_0,x,t)|^p\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0\\
&\le \int_{\partial\Omega}|u(r_0,x,t)|^p\,\mathrm{d}x\,\mathrm{d}t + \int_{\partial\Omega}|u(0,x,t)|^p\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + C_2\left(\|\mu_1\|_{C,2r}+\|\mu_2\|_{C,2r}+\|\mu_2\|^{1/2}_{C,2r}\right)\|N^r(u)\|^p_{L^p(\partial\Omega)}.
\end{aligned}$$

Finally truncating the first integral on the left hand side to $[0,r_0/2]$ gives

$$\begin{aligned}
&\frac{C_1}{2}\int_0^{r_0/2}\int_{\partial\Omega}|\nabla u|^2|u|^{p-2}x_0\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0 + \frac{2}{r_0}\int_0^{r_0}\int_{\partial\Omega}|u(x_0,x,t)|^p\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0\\
&\le \int_{\partial\Omega}|u(r_0,x,t)|^p\,\mathrm{d}x\,\mathrm{d}t + \int_{\partial\Omega}|u(0,x,t)|^p\,\mathrm{d}x\,\mathrm{d}t \qquad (4.5.8)\\
&\quad + C_2\left(\|\mu_1\|_{C,2r}+\|\mu_2\|_{C,2r}+\|\mu_2\|^{1/2}_{C,2r}\right)\|N^r(u)\|^p_{L^p(\partial\Omega)}.
\end{aligned}$$

The local estimate for lemma 4.5.1 is obtained (exactly as in [DH18]) if we do not sum over all the coordinate patches but instead use the estimates obtained for a single boundary cube Q_r in (4.5.5). The $\mathrm{Lip}(1,1/2)$ norm ℓ comes from the pullback mapping and the deformation of the cubes. □

Remark 4.5.3 (Finiteness of the p-adapted square function)**.** We cannot currently assume the a priori finiteness of $S_{p,a}(u)$ and $\|S_{p,a}(u)\|_{L^p}$ so we need to justify how the steps taken for the above proof can be adjusted to account for this. The idea is to remove an ε strip in the x_0 direction, use theorem 4.4.3 for the finiteness of the integrals (since these are now interior integrals) and then use an approximating argument and the monotone convergence theorem.

To perform this argument we replace (4.5.3) by

$$\int_{\varepsilon}^{r} \int_{Q_{2r}} |u|^{p-2} \frac{a_{ij}}{a_{00}} (\partial_i u)(\partial_j u) \zeta^2 (x_0 - \varepsilon) \, \mathrm{d}x \, \mathrm{d}t \, \mathrm{d}x_0,$$

which removes an ε strip in the x_0 direction. Theorem 4.4.3 then justifies that this expression is finite and performing the same steps for the local estimate we obtain

$$\int_{T(Q_r)\setminus\{x_0\le\varepsilon\}} |\nabla u|^2 |u|^{p-2} (x_0 - \varepsilon) \, \mathrm{d}x_0 \, \mathrm{d}x \, \mathrm{d}t \le C(1 + \|\mu\|_{C,r_0})(1+\ell^2) \int_{Q_{2r}} \left(N^{2r}\right)^2 (u) \, \mathrm{d}x \, \mathrm{d}t.$$

Letting $\varepsilon \to 0$ and applying the monotone convergence theorem gives us the desired results lemmas 4.5.1 and 4.5.2. Therefore via Tonelli's theorem $S_{p,a}(u)$ and $\|S_{p,a}(u)\|_{L^p}$ are finite and hence so is the p-adapted area function.

We finally need to control the first integral on the right hand side of (4.5.2) to achieve our goal of controlling the p-adapted square function. Thankfully this has already been done for us in the proof of [DH18, Cor. 5.3] which we encapsulate below.

Lemma 4.5.4. *Let Ω be as in lemma 4.5.2 and u be a non-negative solution to* (4.1.1)*. For a small $r_0 > 0$ depending on the geometry the domain Ω there exists a constant C such that for $\varepsilon = \|\mu_1\|_{C,2r} + \|\mu_2\|_{C,2r} + \|\mu_2\|_{C,2r}^{1/2}$*

$$\int_{\partial\Omega} u(r_0, x, t)^p \, \mathrm{d}x \, \mathrm{d}t \le \frac{2}{r_0} \int_0^{r_0} \int_{\partial\Omega} u(x_0, x, t)^p \, \mathrm{d}x \, \mathrm{d}t \, \mathrm{d}x_0 + C\varepsilon \|N^{r_0}(u)\|_{L^p(\partial\Omega)}^p.$$

Combining lemmas 4.5.2 and 4.5.4 gives us the desired result.

Corollary 4.5.5. *Let Ω be as in lemma 4.5.2 and u be a non-negative solution to* (4.1.1)*. For a small $r_0 > 0$ depending on the geometry the domain Ω there exists constants $C_1, C_2 > 0$ such that for $\varepsilon = \|\mu_1\|_{C,2r} + \|\mu_2\|_{C,2r} + \|\mu_2\|_{C,2r}^{1/2}$*

$$\begin{aligned} \|S_p^{r_0/2}(u)\|_{L^p(\partial\Omega)}^p &\sim \int_0^{r_0/2} \int_{\partial\Omega} |\nabla u|^2 |u|^{p-2} x_0 \, \mathrm{d}x \, \mathrm{d}t \, \mathrm{d}x_0 \\ &\le C_1 \int_{\partial\Omega} |u(0,x,t)|^p \, \mathrm{d}x \, \mathrm{d}t + C_2 \varepsilon \|N^{r_0}(u)\|_{L^p(\partial\Omega)}^p. \end{aligned} \tag{4.5.9}$$

4.6 Bounding the Non-tangential Maximal Function by the p-adapted Square Function

Our goal in this section has been vastly simplified due to Rivera-Noriega [Riv03] proving a local good-λ inequality. We use this to bound the non-tangential maximal function by the p-adapted square function. We first bound the non-tangential maximal function by the usual L^2 based square function $S_2(u)$ but a simple argument from [DPP07, (3.41)] shows that for $1 < p < 2$ and any $\varepsilon > 0$ we have

$$\|S_2^r(u)\|_{L^p(\partial\Omega)} \le C_\varepsilon \|S_p^r(u)\|_{L^p(\partial\Omega)} + \varepsilon \|N^r(u)\|_{L^p(\partial\Omega)}, \tag{4.6.1}$$

with a local version of this statement holding as well.

The good–λ inequality from [Riv03, p. 508] is expressed in the following lemma, where we've used the pullback PDE.

Lemma 4.6.1. *Let v be a solution to* (4.2.54) *and $v(X,t) = 0$ at a point $(X,t) \in Q_r = Q_r(y,s)$. If $E(\lambda) = \{(0,x,t) \in Q_r : S_{2,a}(v) \le \lambda\}$ and $q > 2$ then*

$$|\{(0,x,t) \in Q_r : N_a(v) > \lambda\}| \lesssim |\{(0,x,t) \in Q_r : S_{2,a}(v) > \lambda\}| + \frac{1}{\lambda^q} \int_{E(\lambda)} S_{2,a}(v)^q \, \mathrm{d}x \, \mathrm{d}t. \tag{4.6.2}$$

If $p \geq 2$ then the following lemma is immediate from [DH18, Lemma 6.1], which is an adaptation of [Riv03, Theorem 1.3 and Proposition 5.3].

Lemma 4.6.2. *Let v be a solution to* (4.2.54) *in U and if the coefficients of* (4.2.54) *satisfy the Carleson estimates* (4.2.63), (4.2.64), (4.2.66) *and* (4.2.67) *on all parabolic cubes of size $\leq r_0$ then there exists a constant C such that for any $r \in (0, r_0/8)$*

$$\int_{Q_r} N_{a/12}(v)^p \,\mathrm{d}x\,\mathrm{d}t \leq C\left(\int_{Q_{2r}} A_{2,a}(v)^p \,\mathrm{d}x\,\mathrm{d}t + \int_{Q_{2r}} S_{2,a}(v)^p \,\mathrm{d}x\,\mathrm{d}t\right) + r^{n+1}|v(V_r)|^p, \quad (4.6.3)$$

where V_r is a corkscrew point of the boundary cube Q_r.

Proof. We first assume that $v(X,t) = 0$ for some $(X,t) \in Q_r$ and then we have the good-λ inequality (4.6.2). The passage from this good-λ inequality to a local L^p estimate is usually stated as being 'standard in the spirit of [FS72]'. However we include it here for completeness. First note using that $S_{2,a}(u) \leq \lambda$ on $E(\lambda) = \{(0,x,t) \in Q_r : S_{2,a}(v) \leq \lambda\}$ and we can bound the integral in (4.6.2) as

$$\int_{E(\lambda)} S_{2,a}(u)^q \,\mathrm{d}x\,\mathrm{d}t = \int_0^\lambda \alpha^{q-1}|\{(0,x,t) \in Q_r : S_{2,a}(u) > \alpha\}| \,\mathrm{d}\alpha.$$

Let $p < q$, multiply (4.6.2) by $p\lambda^{p-1}$ and then integrate in λ over 0 to ∞ giving

$$\begin{aligned}
\|N_a(u)\|^p_{L^p(Q_r)} &\lesssim \|S_{2,a}(u)\|^p_{L^p(Q_{2r})} + p\int_0^\infty \lambda^{p-1-q}\int_0^\lambda \alpha^{q-1}|\{(0,x,t) \in Q_r : S_{2,a}(u) > \alpha\}| \,\mathrm{d}\alpha\,\mathrm{d}\lambda \\
&= \|S_{2,a}(u)\|^p_{L^p(Q_{2r})} + p\int_0^\infty \alpha^{q-1}|\{(0,x,t) \in Q_r : S_{2,a}(u) > \alpha\}| \int_\alpha^\infty \lambda^{p-1-q} \,\mathrm{d}\lambda\,\mathrm{d}\alpha \\
&= \|S_{2,a}(u)\|^p_{L^p(Q_{2r})} + p(q-p)\int_0^\infty \alpha^{p-1}|\{(0,x,t) \in Q_r : S_{2,a}(u) > \alpha\}| \,\mathrm{d}\alpha \\
&\lesssim \|S_{2,a}(u)\|^p_{L^p(Q_{2r})}.
\end{aligned}$$

We remove the assumption $v(X,t) = 0$ for the cost of adding the area function and $r^{n+1}|v(V_r)|^p$ term in the same way as [Riv03; DH18]. □

From this local estimate we can obtain the following global L^p estimate by the same proof as the global L^2 estimate from [DH18, Theorem 6.2], which is based on [DPP07, Proposition 3.2].

Theorem 4.6.3. *Let u be a solution to* (4.1.1) *and the coefficients of* (4.1.1) *satisfy the Carleson estimates* (4.2.65) *and* (4.2.68) *then*

$$\|N^r(u)\|_{L^p(\partial\Omega)} \lesssim \|S_2^r(u)\|_{L^p(\partial\Omega)} + \|u\|_{L^p(\partial\Omega)} \quad (4.6.4)$$

and by (4.6.1)

$$\|N^r(u)\|_{L^p(\partial\Omega)} \lesssim \|S_p^r(u)\|_{L^p(\partial\Omega)} + \|u\|_{L^p(\partial\Omega)}. \quad (4.6.5)$$

We include an outline of the proof from [DH18] adapted to the L^p setting here.

Proof. We first prove the following variation of lemma 4.6.2

$$\int_{Q_r} N_{a/12}(u)^p \,\mathrm{d}x\,\mathrm{d}t \lesssim \int_{Q_{2r}} S_{2,a}(u)^p \,\mathrm{d}x\,\mathrm{d}t + \int_{Q_{2r}} A_{2,a}(u)^p \,\mathrm{d}x\,\mathrm{d}t \quad (4.6.6)$$

in the subspace

$$\mathcal{U}_0 = \left\{u : \int_{Q_r} u \,\mathrm{d}x\,\mathrm{d}t = 0\right\}.$$

This is proved by contradiction. If (4.6.6) does not hold then there exists a sequence $u_k \in \mathcal{U}_0$

such that

$$\int_{Q_r} N_{a/12}(u)^p \,\mathrm{d}x\,\mathrm{d}t = 1, \tag{4.6.7}$$

$$\int_{Q_{2r}} S_{2,a}(u)^p \,\mathrm{d}x\,\mathrm{d}t \leq 1/k, \quad \text{and} \quad \int_{Q_{2r}} A_{2,a}(u)^p \,\mathrm{d}x\,\mathrm{d}t \leq 1/k. \tag{4.6.8}$$

Equation (4.6.7) implies for any point $(y_0, y, s) \in \Gamma_{a/12}(x, t)$ and $(x, t) \in Q_r$ then

$$|u_k(y_0, y, s)| \leq C(y_0),$$

for some constant $C(y_0)$, which blows up as $y_0 \to 0_+$. The Arzelà-Ascoli theorem says we can find a subsequence u_{k_j} that converges locally uniformly to a function u on all compact subsets K of the union of cones $\Gamma_{a/12}(x, t)$ for $(x, t) \in Q_r$. On any such K by (4.6.8) the full gradient $Du_{k_j} \to 0$ and therefore u_{k_j} converges to a constant function u on the union of all cones $\Gamma_{a/12}(x, t)$ for $(x, t) \in Q_r$.

Furthermore, without too much additional work and using lemma 4.6.2 one can show u has average 0 and hence $u = 0$. On the other hand, by taking appropriate limits in (4.6.7)

$$\int_{Q_r} N_{a/12}(u)^p \,\mathrm{d}x\,\mathrm{d}t = 1$$

giving a contradiction. Therefore on the subspace $\mathcal{U}_0$ equation (4.6.6) holds.

To pass to the general case let $v = u - \fint_{Q_r} u \,\mathrm{d}x\,\mathrm{d}t$ then $v \in \mathcal{U}_0$ and by applying (4.6.6)

$$\int_{Q_r} N_{a/12}(u)^p \,\mathrm{d}x\,\mathrm{d}t \lesssim \int_{Q_{2r}} S_{2,a}(u)^p \,\mathrm{d}x\,\mathrm{d}t + \int_{Q_{2r}} A_{2,a}(u)^p \,\mathrm{d}x\,\mathrm{d}t + \left(\int_{Q_r} u(x, t) \,\mathrm{d}x\,\mathrm{d}t \right)^p.$$

To conclude the global estimate we apply Hölder to the last term, control the area function by the square function using lemma 4.4.10, and then sum over all parabolic cubes Q_r covering $\partial\Omega$. □

4.7 Proof of Theorem 4.1.6

We only consider the case $1 < p < 2$ and use interpolation to obtain solvability for $p \geq 2$. First assume either stronger Carleson condition of (4.2.68), or (4.1.7) and (4.1.8) hold. Therefore the Carleson conditions on the pullback coefficients (4.2.63), (4.2.64), (4.2.66) and (4.2.67) hold.

Without loss of generality, by remark 4.2.28, we may assume that our domain is smooth. Consider $f^+ = \max\{0, f\}$ and $f^- = \max\{0, -f\}$ where $f \in C_0(\partial\Omega)$ and denote the corresponding solutions with these boundary data u^+ and u^- respectively. Hence we may apply the corollary 4.5.5 separately to u^+ and u^-. By the maximum principle these two solutions are non-negative. It follows that for any such non-negative u we have

$$\|S_p^r(u)\|^p_{L^p(\partial\Omega)} \leq C\|f\|^p_{L^p(\partial\Omega)} + C\left(\|\mu\|^{1/2}_{C,r_0} + \|\mu\|_{C,r_0}\right)\|N^{2r}(u)\|^p_{L^p(\partial\Omega)}$$

and theorem 4.6.3 gives

$$\|N^r(u)\|^p_{L^p(\partial\Omega)} \leq C\|f\|^p_{L^p(\partial\Omega)} + C\|S_p^{2r}(u)\|^p_{L^p(\partial\Omega)},$$

here $\|\mu\|_{C,r_0}$ is the Carleson norm of (4.1.7) on Carleson regions of size $\leq r_0$. As noted earlier, if for example Ω is of VMO-type then the size of μ appearing in this estimate only depends on the Carleson norm of coefficients on Ω, provided we only consider small Carleson regions. Hence we can choose r_0 small enough (depending on the domain Ω) such that the Carleson norm after the pullback is say only twice the original Carleson norm of the coefficients over all cubes of size $\leq r_0$.

Since we are assuming $\|\mu\|_{C,r_0}$ is small, clearly we also have $\|\mu\|_{C,r_0} \leq C\|\mu\|^{1/2}_{C,r_0}$. By rearranging these two inequalities and combining estimates for u^+ and u^-, we obtain, for

$0 < r \le r_0/8$,

$$\|N^r(u)\|^p_{L^p(\partial\Omega)} \le C\|f\|^p_{L^p(\partial\Omega)} + C\|\mu\|^{1/2}_{C,r_0}\|N^{4r}(u)\|^p_{L^p(\partial\Omega)}.$$

By a simple geometric argument in [DH18] involving cones of different apertures, lemmas 2.2.4 and 2.4.4 show that there exists a constant M such that

$$\|N^{4r}(u)\|^p_{L^p(\partial\Omega)} \le M\|N^r(u)\|^p_{L^p(\partial\Omega)}. \tag{4.7.1}$$

It follows that if $CM\|\mu\|^{1/2}_{C,r_0} < 1/2$ then by combining the last two inequalities we obtain

$$\|N^r(u)\|^p_{L^p(\partial\Omega)} \le 2C\|f\|^p_{L^p(\partial\Omega)},$$

which is the desired estimate (for truncated version of non-tangential maximal function). The result with the non-truncated version of the non-tangential maximal function $N(u)$ follows as our domain is bounded in space and hence (4.7.1) can be iterated finitely many times until the non-tangential cones have sufficient height to cover the whole domain.

Finally, we comment on how the Carleson condition (4.2.68) can be relaxed to the weaker condition (4.1.6). The idea is the same as [DH18, Theorem 3.1; DPP17, pp. 1178–1179]. As shown there, if the operator $L = \operatorname{div}(A\nabla) + B\cdot\nabla - \partial_t$ satisfies the weaker condition (4.1.6), then it is possible (via mollification of coefficients) to find another operator L_1 which is a small perturbation of the operator L and L_1 satisfies (4.2.68). The solvability of the L^p Dirichlet problem in the range $1 < p < 2$ for L_1 follows by our previous arguments. However, as L is a small perturbation of the operator L_1 we have by the perturbation arguments of [Swe98; Nys97] in theorem 2.4.33 L^p solvability of L as well.

For larger values of p we use the maximum principle and interpolation to obtain solvability results in the full range $1 < p \le \infty$. □

Chapter 5

Open Problems

- Do we have the equivalence between vanishing Carleson measures and VMO when ψ is not compactly supported but instead satisfies some nice decay estimates? See remark 2.5.15 for details.

- In section 4.2 we obtain equivalent conditions for $\mathbb{D}\phi \in \mathrm{BMO}$, one of which we used to localise this condition in defining admissible domains. Do our VMO-type domains imply that $\mathbb{D}\phi \in \mathrm{VMO}$? Or does the condition: for all $\eta > 0$ there exists $r_1 > 0$ such that

$$\sup_{\substack{Q_s = J_s \times I_s \\ Q_s,\, s \le r_1}} \frac{1}{|Q_s|} \int_{Q_s} \int_{I_s} \frac{|\phi(x,t) - \phi(x,\tau)|^2}{|t-\tau|^2} \,\mathrm{d}\tau \,\mathrm{d}t \,\mathrm{d}x \le \eta^2 \tag{5.0.1}$$

  and $\nabla\phi \in \mathrm{VMO}$ imply $\mathbb{D}\phi \in \mathrm{VMO}$ (first globally and then locally in a domain)? This would justify us calling them VMO-type domains in definition 4.2.20.

  More generally do theorems 4.2.5 and 4.2.7 in section 4.2 hold for VMO instead of BMO? This might already be a trivial consequence of the work that we have done and dyadic harmonic analysis. However this area of mathematics is not an area we are familiar with. See [Vil15; PPV17; OV17] and references therein for a start in this direction looking at endpoint properties of Calderòn-Zygmund operators via dyadic harmonic analysis.

- From remark 4.3.4: instead of assuming the smallness condition of B in (1.0.3), $\delta|B| \le K$, can we instead replace the smallness condition with smallness in the Carleson norm of B in either (4.1.6) or (4.1.7)?

- Show the $(R)_p$ property via a Carleson condition on coefficients, as in chapter 4. This was done in the elliptic case in [DPR17] however we had difficulties generalising this to the parabolic setting. We tried to prove the $N < S$ bound, c.f. section 4.5, but we obtained some strange terms during the integration by parts:

$$\int_U |u_t|^2 x_0 \,\mathrm{d}x_0 \,\mathrm{d}x \,\mathrm{d}t. \tag{5.0.2}$$

  If one compares this to the area function (4.4.4) when $p = 2$ one can see that the x_0^3 has been replaced by x_0.

  We believe that this is closely related to terms that will appear with similar scaling when one is studying the elliptic regularity problem for sets with higher co-dimensional boundaries, see [DFM17c; DFM17b; DFM17a][1]. Since we are in a slightly more usual and understood setting, a solution to these difficulties might be easier to obtain here and then transfer over to the elliptic case with higher co-dimensional boundaries. In any case,

[1] There is a different notion of ellipticity in this setting to counteract the lower co-dimensional boundary. “In some sense, we create Brownian travellers that treat Γ [the boundary] as a “black hole”: they detect more mass and they are more attracted to Γ than a standard Brownian traveller governed by the Laplacian would be.” [DFM17c]

if a solution is discovered in the higher co-dimensional boundary case then it should be possible to transfer it to the parabolic setting.

- $(R)_p \implies (D)_{p^*}$, where p^* is an exponent depending on the dimension as in the elliptic system case [She06]. This is proved via a reverse Hölder inequality for boundary data vanishing on a ball. It is also studied for complex coefficient elliptic equations in [DP18a, Theorem 1.2].
- Perturbation of $(R)_p$ results. Does $(R)_p \implies (R)_{p-\varepsilon}$ for some ε? Does $(R)_p \implies (R)_q$ for $1 < q < p$? Both of these results are shown in [KP93] in the elliptic setting.
- Does $(R)_q + (D^*)_{p'} \implies (R)_p$ for $1 < q < p$? See [She07; DK12] for the elliptic proof of this.
- Endpoint regularity problem. Define parabolic HS^1 for $(R)_1$ and then show the perturbation results for the regularity problem. Is there an atomic decomposition for parabolic HS^1? Does $(R)_p \implies (R)_1$ for all $p > 1$? Does $(R)_1 + (D^*)_{p'} \implies (R)_p$? See [KP93; She07; DK12] for the results in the elliptic setting and [Bro90] for the endpoint space in the parabolic case, which might be the appropriate parabolic HS^1 space for the parabolic regularity problem.
- Complex coefficient results for the parabolic L^p Dirichlet problem. Are we able to extend the recent work of Dindoš and Pipher [DP16; DP18b; DP18a] from the elliptic to the parabolic setting?
- We could extend theorem 3.1.1 in chapter 3 of this thesis to include a divergence free drift term satisfying a suitable smallness condition (on Lewis-Murray cylinders). In this case the adjoint of (1.0.1) would be $-u_t = \operatorname{div}(A^*\nabla u) + \operatorname{div}(B\nabla u)$ which after reflecting in time is the same form as (1.0.1). However, to streamline the presentation and notation we chose not to include this minor extension. See comments in [HL01] about divergence free drift terms.

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